LOGICS ON PARTITIONS

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Information is context-dependent. Moreover, our life is more complicated: Ambiguity of a message suggests that even in a single situation more than one context can exist.

Taking it for granted that a partition of possible worlds serves as a representation of a context, the idea of multi-contextual situation motivates the proposal of partitionistic structures in this paper.

The chapter aims to investigate logics arising from partitionistic structures. In addition to completeness and finite model property of the basic partition logic and some characterization results, it is shown that the class of p-frames with linearly ordered partitions cannot be characterized.

1. Language and semantics

The modal language here has modal operators $[\forall]$ and ∇.

A p-frame is $\langle W, \Pi \rangle$, where W is an non-empty set, and Π is a set of partitions on W, i.e. each $P \in \Pi$ is a partition of W. A valuation v on p-frame $F = \langle W, \Pi \rangle$ is a function which assigns a subset of W to a propositional letter. A p-model on a p-frame $F = \langle W, \Pi \rangle$ is $\langle W, \Pi, v \rangle$, where v is a valuation on F.

Let $[x]_P$ stand for the member $U \in P \in \Pi$ such that $x \in U$.

The truth condition for ∇ is: $M, x \models \nabla A$ iff there is $P \in \Pi$ such that $[x]_P$ is the truth set of A in M; and the truth condition for $[\forall]$ is: $M, x \models [\forall]A$ iff $M, w \models A$ for every world $w \in W$. When ∇A holds at x in M, a *witness* for ∇A at x in M is $P \in \Pi$ such that $[x]_P \in P$. It may not be unique.

Substitution to propositional variables works as usual.

2. Comparison with neighborhood semantics

Segerberg [6] investigates neighborhood semantics of modal logic. A neighborhood frame F is a pair $\langle U, \nu \rangle$, where U is a set and ν is a function which assigns to $x \in U$ a set of subsets of U; a truth value

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assignment v on F is a function from propositional variables to the powerset of U; a neighborhood model on F is a pair $\langle F, v\rangle$ where v is a truth value assignment on F; and the notion is recursively defined as usual; in particular, $\star A$ is true at $x \in U$ iff there is $s \in \nu(x)$ such that s is the truth set of A.

A neighborhood frame F is *reflexive* if for every $x \in U$ if $\nu(x) \neq \emptyset$ then $x \in \bigcap \nu(x)$. It is easy to observe that F is reflexive iff the scheme $\star p \to p$ is valid in F. In fact, for each p-frame (or each p-model), a reflexive neighborhood frame (model) can be defined as follows. Given a p-frame $F_p = \langle W, \Pi\rangle$, define a neighborhood frame $N(F_p) = \langle W, \nu\rangle$ where the neighborhood set $\nu(x)$ of x is defined as $\nu(x) = \{[x]_P : P \in \Pi\}$. Take v on the p-frame as a valuation on the corresponding neighborhood frame. Then:

Proposition 1. *$F_p, v, x \models A$ iff $N(F_p), v, x \models A$.*

In particular, when $\Pi \neq \emptyset$, $\nu(x) \neq \emptyset$ for each x.

There is no reverse translation; for the class of all p-frames is a subclass of the class of all reflexive neighborhood frames, with any neighborhood sets for each point are nested. Thus a stronger logic is expected; is there a strictly stronger logic? The answer is yes.

3. Basic partition logic: axiomatic system $L([\forall], \nabla)$

(1) PC
(2) S5 for $[\forall]$
(3) $[\forall](p \to \nabla p) \vee [\forall](p \to \neg\nabla p)$
(4) $\nabla p \to p$ (Reflexivity)

Soundness is easy. The following are deducible from the above:

- $[\forall]\nabla\top \vee [\forall]\neg\nabla\top$
- $\langle\exists\rangle\nabla A \to [\forall](\nabla A \leftrightarrow A)$
- $\langle\exists\rangle(A \wedge \neg\nabla A) \to [\forall]\neg\nabla A$
- $[\forall](p \to q) \to ([\forall](\nabla p \to \nabla q) \vee [\forall](\nabla p \to \neg\nabla q))$
- $[\forall](A \leftrightarrow B) \to [\forall](\nabla A \leftrightarrow \nabla B)$
- $(\langle\exists\rangle\nabla B \wedge \langle\exists\rangle\nabla\neg B) \to [\forall](\nabla B \vee \nabla\neg B)$

If X is a finite set of formulas, $\bigwedge X$ is the conjunction of all formulas in X, and $\bigvee X$ is the disjunction of them. (Let $\bigwedge\emptyset = \top$ and $\bigvee\emptyset = \bot$ conventionally.)

4. Completeness and finite model property

Completeness and finite model property are shown simulteneously by the following construction of a finite model of a given consistent formula, where each point is a formula.

Let A be a consistent formula. Pick a maximal consistent set X_A such that $A \in X_A$. Let Σ be the set of maximal consistent sets Y such that $[\forall]^- X_A = \{B : [\forall]B \in X_A\} \subseteq Y$.

Let $Prop_A$ be the set of propositional variables occurring in A. For the set $Sub(A)$ of all subformulas of A, define:

$$
\begin{array}{rcl}
\Gamma_0 & = & Sub(A) \cup \{\neg B : B \in Sub(A)\} \\
\Gamma_1 & = & \Gamma_1 \cup \{\nabla B : B \in \Gamma_0\} \\
\Gamma_2 & = & \Gamma_2 \cup \{\neg[\forall]\neg B : B \in \Gamma_1\} \\
\Gamma_A & = & \bigcup_{B \in \Gamma_2} Sub(B)
\end{array}
$$

Also define for each $Y \in \Sigma$,

$$
\begin{array}{rcl}
D(Y) & = & \Gamma_A \cap Y \\
C(Y) & = & \bigwedge D(Y)
\end{array}
$$

Let $\Omega = \{B \in Sub(A) : \neg[\forall]\neg\nabla B \in X_A\}$.

Proposition 2. *Let $B \in \Omega$ and $B' \in Sub(A)$. If $[\forall](B \leftrightarrow B') \in X_A$ then $B' \in \Omega$.*

Proof. It holds because $[\forall](A \leftrightarrow B) \rightarrow [\forall](\nabla A \leftrightarrow \nabla B)$ is a theorem. □

Let $U_0 = \{C(Y) : Y \in \Sigma\}$. Since Γ_A is finite, so is U_0.

Observation 3. *From the definitions,*

- *For any $C \in U_0$, $C \nvdash \bot$.*
- *For any $C \in U_0$, $\langle\exists\rangle C \in X_A$.*
- *There is a unique C among C_i's such that $C \in X_A$.*
- *For any $B \in Sub(A)$, if $\langle\exists\rangle B \in X_A$, there exists $C \in U_0$ such that $C \vdash B$.*
- *For any $C \in U_0$, and for any B on $Prop_A$, if $C \vdash [\forall]B$, then $[\forall]B \in X_A$.*
- *If $C \neq C'$, $\vdash C \wedge C' \leftrightarrow \bot$.*
- *$C \vdash \langle\exists\rangle C'$ for any $C, C' \in U_0$.*
- *Let $E_A = \bigwedge(\Gamma_A \cap [\forall]^- X_A)$. Then*
  - *$C \vdash E_A$ for each $C \in U_0$.*
  - *$\vdash \bigvee_{C \in U_0} C \leftrightarrow E_A$.*

Classify Ω as follows:

$$
\begin{array}{rcl}
O^\top & = & \{B \in \Omega : [\forall]\nabla B \in X_A\} \\
O^+ & = & \{B \in \Omega : \langle\exists\rangle\nabla\neg B \in X_A\} \\
O^- & = & \{B \in \Omega : \neg[\forall]\nabla B \wedge [\forall]\neg\nabla\neg B \in X_A\}
\end{array}
$$

These sets are mutually disjoint and exhaust Ω.

Intuitively, formulas in O^+ and $O^\top$ behave well, while those in O^- require taming.

Proposition 4. *Properties of formulas in O^+ and $O^\top$: For each $B \in \Omega$,*

- *$B \in O^\top$ iff for each $C \in U_0$, $C \vdash [\forall]\nabla B$.*
- *Suppose $B \in O^\top$. Then $[\forall]B \in X_A$.*
- *If $B \in O^+$, then there exists $C \in U_0$ such that $C \vdash B$, there exists C' such that $C' \vdash \neg B$. Moreover, for any $C \in U_0$, $C \vdash B$ or $C \vdash \neg B$.*

Proof. Note that all formulas in this argument are on $Prop_A$, and that in particular every $p \in Prop_A$ occurs in C.

- Right to left is obvious. Suppose $B \in O^\top$. Then as $[\forall]\nabla B \in X_A$, for every $C \in U_0$, $C \vdash [\forall]\nabla B$.
- By theorem $\langle\exists\rangle\nabla B \to [\forall](\nabla B \leftrightarrow B)$.
- Suppose $B \in O^+$. Then $\langle\exists\rangle\nabla B \in X_A$ and $\langle\exists\rangle\nabla\neg B \in X_A$. Thus, there exists $C \in U_0$ such that $C \vdash B$ and there exists $C' \in U_0$ such that $C' \vdash \neg B$.

  As $(\langle\exists\rangle\nabla B \wedge \langle\exists\rangle\nabla\neg B) \to [\forall](\nabla B \vee \nabla\neg B)$ is a theorem, $[\forall](\nabla B \vee \nabla\neg B) \in X_A$, so $C_i \vdash \nabla B \vee \nabla\neg B$ for each C_i. Moreover, since $\langle\exists\rangle\nabla A \to [\forall](\nabla A \leftrightarrow A)$ is a theorem, $C \vdash \nabla B \leftrightarrow B$ and $C \vdash \nabla\neg B \leftrightarrow \neg B$ for each C for every $C \in U_0$.
  - Suppose $C \not\vdash B$. Then, $C \not\vdash \nabla B$. Thus $C \vdash \nabla\neg B$ which entails $C \vdash \neg B$.
  - Suppose $C \not\vdash \neg B$. Then, $C \not\vdash \nabla\neg B$. Thus $C \vdash \nabla B$ which entails $C \vdash B$.

  Thus, for each C, $C \vdash B$ or $C \vdash \neg B$.

□

Let $U_f = U_0 \times \{0\} \cup U_0 \times \{1\}$. Introduce partitions on U_f for each $B \in \Omega_A$ as follows:

- $B \in O^\top$: $P_B = \{U_f\}$.
- $B \in O^+$ $P_B = \{\{\langle C, j\rangle :\vdash C \to B, j = 0.1\}, \{\langle C, j\rangle :\not\vdash C \to B, j = 0.1\}\}$
- $B \in O^-$:
  $P_B = \{\{\langle C, j\rangle :\vdash C \to B, j = 0.1\}, \{\langle C, 0\rangle :\not\vdash C \to B\}, \{\langle C, 1\rangle :\not\vdash C \to B\}\}$

Let $\Pi_f = \{P_B : B \in \Omega\}$.

Proposition 5. *Let $B \in Sub(A)$. There is $P \in \Pi_f$ such that $\{\langle C, j\rangle :\vdash C \to B, j = 0.1\} \in P$ iff $B \in \Omega$.*

Proof. Right to left is by definition. Suppose there is $P \in \Pi_f$ such that $\{\langle C, j\rangle :\vdash C \to B, j = 0.1\} \in P$. Then, there is $B' \in \Omega$ such that There is $P \in \Pi_f$ such that $\{\langle C, j\rangle :\vdash C \to B', j = 0.1\} \in P$. By Proposition 2, $B \in \Omega$. □

Define $v = f : Prop \to \wp U_f$ as: $v_f(p) = \{\langle C, j\rangle : C \vdash p\}$ if $p \in Prop_A$, and $v_f(p) = \emptyset$ otherwise.

Define $M_f = \langle U_f, \Pi_f, v_f\rangle$. Then M_f is a p-model.

Lemma 6. *Let $B \in Sub(A)$. Then $M_f, \langle C, j\rangle \models B$ iff $C \vdash B$.*

Proof. By induction on B.

- B is a propositional variable. $M_f, \langle C, j\rangle \models p$ iff $\langle C, j\rangle \in v_f(p)$ iff $C \vdash p$.
- A standard argument works when $B = B_0 \wedge B_1$ or $B = \neg B_0$.
- When $B = [\forall]B_0$. $M_f, \langle C, j\rangle \models [\forall]B_0$ iff for every $\langle C', j'\rangle$ $M_f, \langle C', j'\rangle \models B_0$ iff for every $\langle C', j'\rangle$ $C' \vdash B_0$ iff $\bigvee_{C' \in U_0} C' \vdash B_0$ iff $E_A \vdash B_0$ iff $E_A \vdash [\forall]B_0$ iff $C \vdash [\forall]B_0$.
- $B = \nabla B_0$. Suppose $C \vdash \nabla B_0$. Then $\langle\exists\rangle\nabla B_0 \in X_A$, so $B_0 \in \Omega$. By definition of Π_f, for any member of Ω, there is $P \in \Pi_f$ such that $\langle C, j\rangle \in \{\langle C', j'\rangle : C' \vdash B_0\} \in P$. By induction hypothesis, it is equivalent to that there is $P \in \Pi_f$ such that $\langle C, j\rangle \in \{\langle C', j'\rangle : M_f, \langle C', j'\rangle \models B_0\} \in P$, which is equivalent to $M_f, \langle C, j\rangle \models \nabla B_0$.

  Conversely, suppose $M_f, \langle C, j\rangle \models \nabla B_0$. It is equivalent to that there is $P \in \Pi_f$ such that $\langle C, j\rangle \in \{\langle C', j'\rangle : M_f, \langle C', j'\rangle \models B_0\} \in P$, which in turn is equivalent by induction hypothesis to that there is $P \in \Pi_f$ such that $\langle C, j\rangle \in \{\langle C', j'\rangle : C' \vdash B_0\} \in P$.

  For such a set is in P, $B_0 \in \Omega$ must hold by Proposition 5.
  - $B_0 \in O^\top$: Then by Proposition 4 for any $C \vdash [\forall]B_0$, and moreover, as $\langle\exists\rangle\nabla B_0 \to [\forall](B_0 \leftrightarrow \nabla B_0)$ is a theorem, $C \vdash \nabla B_0$.
  - $B_0 \in O^+$: Then there exists C such that $C \vdash B$, there exists C such that $C \vdash \neg B$, and for any C, $C \vdash B$ or $C \vdash \neg B$. As $C \vdash B_0$, and $[\forall](B_0 \leftrightarrow \nabla B_0)$, $C \vdash \nabla B_0$.
  - $B_0 \in O^-$: $C \vdash B$ and $[\forall](B_0 \leftrightarrow \nabla B_0)$, $C \vdash \nabla B_0$.

  Thus, in any cases, $C \vdash \nabla B_0$.

Therefore, Then $M_f, \langle C, j\rangle \models B$ iff $C \vdash B$. □

From the argument above,

Theorem 7. *The class of all p-frames is characterized by the system of basic partition logic.*

5. Regular partition logics

The notions of regular and normal p-frames are similar to Segerberg [6]'s on neighborhood frames.

For $F = \langle W, \Pi \rangle$ and $x \in W$, let $\Pi(x) = \{U \subseteq W :$ there is $P \in \Pi$ such that $U \in P\}$. A p-frame $F = \langle W, \Pi \rangle$ is *regular* iff Π gives each point a filter, or for each $x \in W$, if $\Pi(x) \neq \emptyset$ then for all $U, V \subseteq W$, $U, V \in \Pi(x)$ iff $U \cap V \in \Pi(x)$.

The logic of the class of all regular p-frames can be axiomatized by:

- Axioms and rules of the basic partition logic
- $\nabla(p \wedge q) \rightarrow \nabla p$
- $\nabla p \wedge \nabla q \rightarrow \nabla(p \wedge q)$
- $\nabla\top \rightarrow (\nabla p \leftrightarrow p)$

The following are deducible:

- $\vdash A \rightarrow B \;/\; \vdash \nabla A \rightarrow \nabla B$
- $\nabla A \rightarrow \nabla\top$
- $\neg\nabla A \vee (\nabla A \leftrightarrow A)$

There are two nice superlogics of the regular partition logic. First, the logic of the class of all p-frames with empty Π. $\neg\nabla p$ characterizes the class[1].

Another special case is the logic of the class of normal p-frames, i.e. p-frames where Π gives each point a non-empty filter. It is axiomatizable with

- Axioms and rules of the regular partition logic
- $\nabla\top$

$\nabla A \leftrightarrow A$ is deducible, i.e. the logic is so-called `Triv`[2].

Those results are observed by considering a relational frame associated to such a structure. As seen before, for any p-frame $F = \langle W, \Pi \rangle$, there is a neighborhood frame $F_n = \langle W, \nu \rangle$ where $\nu(x) = \Pi(x)$. In particular, when F is a regular p-frame, F_n is regular in the sense of neighborhood semantics.

Let $F = \langle W, \nu \rangle$ be regular. The *alternative relation induced by* F is R defined as: Rxy iff x is normal[3] and $y \in \bigcap \nu(x)$. Moreover, a *relational counterpart* of a regular p-frame $F = \langle W, \nu \rangle$ is a relational frame $F_r = \langle W, R, Q \rangle$, where R is the alternative relation induced by F and Q is the set of *singular* points, or $Q = \{x : \nu(x) = \emptyset\}$.

In the case of p-frames, if $\Pi = \emptyset$ then the corresponding $Q = W$; and if $\Pi \neq \emptyset$, then $Q = \emptyset$. A modal sentence $\star A$ is interpreted in a

[1] Segerberg considers a similar case.

[2] Hughes-Cresswell [3, p. 65].

[3] I.e. $\nu(x)$ is a non-empty filter.

regular relational frame as:

$$M, x \models \star A \text{ iff } x \notin Q, \text{ and for all } y(Rxy \Rightarrow M, y \models A)$$

Proposition 8. *Let R be the alternative relation induced by the neighborhood frame F generated from a p-frame $F = \langle W, \Pi \rangle$. Then either $R = \emptyset$, or Rxy implies $x = y$.*

Proof. $\Pi = \emptyset$ implies $Q = W$, which is equivalent to there is no normal point in W. Therefore $Rxy = \emptyset$.

Suppose $\Pi \neq \emptyset$. Then, every $x \in W$ is normal. Take an arbitrary $P \in \Pi$ and distinct $x, y \in W$. Since for every superset S of $[x]_P$ there is $P' \in \Pi$ such that $S \in P'$. In particular, $W \setminus \{y\} \in P'$. For P' to be a partition, $\{y\} \in P'$ must hold. Thus, for every $x \in W$, $\{x\} \in \nu(x)$. Suppose Rxy. By definition, $y \in \bigcap \nu(x)$; in particular $y \in \{x\}$. Therefore, $x = y$. □

Hence, the class of relational counterparts of neighborhood frames generated from empty p-frames $F = \langle W, \emptyset \rangle$ is the class of singular frames. The logic is characterized with $\neg \nabla p$.

On the other hand, the class of relational counterparts of neighborhood frames generated from non-empty p-frames is the class of *normal* frames with the frame condition Rxy implies $x = y$. The logic is thus `Triv`.

6. Singleton partition logic

$(\nabla p \wedge \nabla q) \to [\forall](p \to q)$ characterizes the class of frames where Π is a singleton.

7. Related works

There are several related topics: all-and-only modality and difference modality.

In a relational model $M = \langle W, R, v \rangle$, [ext] is interpreted as $M, x \models [\text{ext}]A$ iff for all y, if NOT Rxy then $M, x \models A$. Humberstone [4] gives an axiomatic system of the logic of valid sentences in the class of all complementary frames with a schema:

(1) K for $\Box$ and [ext]
(2) Let $\mathcal{S}$ be a sequence of $\Box$ and [ext], and $\mathcal{W}$ be the sequence of modalities where each $\star$ in $\mathcal{S}$ is substituted by its dual. Then,

$$\mathcal{W}(\Box p \wedge [\text{ext}]q) \to \mathcal{S}(p \vee q) \tag{1}$$

∇A in singleton frames behaves as $\Box A \wedge [\text{ext}]A$ with the underlying relational S5 frames.

Goranko [2] investigates logics of various classes of complementary frames further to obtain correspondence results; some cases require interaction axioms between $\Box$ and [ext] in addition to characteristic axioms in the standard sense.

De Rijke [1], [5] investigates difference modality: $M, x \models \langle\neq\rangle A$ iff for every $y \neq x$, $M, y \models A$. "$\cdots$ holds only at x" can be defined as $A \wedge \langle\neq\rangle\neg A$. ∇A in singleton frames behaves similarly to "only", in the sense it claims that the proposition holds all and only in the piece of the partition.

8. Linear order on partitions

8.1. **Linear p-frames.** Let W be a set, and Π be a set of partitions of W. Define $\leq_\Pi$ on Π as: $P \leq_\Pi P'$ iff, for any $U \in P$, there is $U' \in P'$ such that $U \subseteq U'$. $\leq_\Pi$ is a partial order.

A *linear p-frame* is a p-frame with Π is linearly ordered by $\leq_\Pi$. A *linear p-model* is a p-model on a linear p-frame.

Consider a system $\mathcal{S}$ which has the following axiom in addition to the axioms and rules of the basic partition logic:

- $(\nabla p \wedge \nabla q) \rightarrow ([\forall](p \rightarrow q) \vee [\forall](q \rightarrow p))$ (Nesting)

8.2. **Impossibility to axiomatize the class of linear p-frames.** $\mathcal{S}$ is sound in the class of all linear p-frames. Moreover, to be shown below, any $\mathcal{S}$-consistent formula has a linear p-model. Nevertheless, $\mathcal{S}$ is sound in a frame $F = \langle W, \Pi\rangle$ where $W = \{a, b, c\}$, $\Pi = \{\{\{a\}, \{b, c\}\}, \{\{a, b\}, \{c\}\}\}$. Clearly F is not a linear p-frame. Thus, the class of all linear p-frames is a proper subclass of that of all $\mathcal{S}$-frames. It means that the class of all linear p-frames cannot be characterized; for, if there were an additional characterizing formula of linear p-frames, it must be falsified in the class of $\mathcal{S}$-frames, so that it cannot be S-theorem, which is absurd.

8.3. **Construction of a linear p-frame.** Now, a linear p-model of a given $\mathcal{S}$-consistent formula A is to be constructed. Some notions and properties must be prepared before the construction.

Let X_A, Γ_A, Ω, and U_0 be defined as before. Let $T = \{V \subseteq U_0 : V \neq \emptyset$ and there is $B \in \Omega$ such that $V = \{C \in U_0 : C \vdash B\}\}$. Then for each non-empty X, $X \in T$ iff there is $B \in Sub(A)$ such that $\{C \in U_0 : C \vdash \nabla B\} = X$.

Proposition 9. *If $\Omega \neq \emptyset$, then $T \neq \emptyset$.*

Proof. Suppose $\Omega \neq \emptyset$. Then there is $B \in Sub(A)$ such that $\langle\exists\rangle\nabla B \in X_A$. It implies that there is $C \in U_0$ such that $C \vdash \nabla B$, and thus $C \vdash B$. $\Box$

Proposition 10. *For all $V, V' \in T$, $V \cap V' \neq \emptyset$ then $V \subseteq V'$ or $V' \subseteq V$.*

Proof. Suppose $V \cap V' \neq \emptyset$, $V = \{C \in U_0 : C \vdash B\}$, and $V = \{C \in U_0 : C \vdash B'\}$. Then there is $C \in U_0$ such that $C \vdash B$ and $C \vdash B'$. Since $\langle\exists\rangle\nabla B \in X_A$ and $\langle\exists\rangle\nabla B' \in X_A$, $[\forall](\nabla B \leftrightarrow B) \in X_A$ and $[\forall](\nabla B' \leftrightarrow B') \in X_A$. Hence, $C \vdash \nabla B \wedge \nabla B'$. By the nesting axiom, $C \vdash [\forall]((B \to B') \vee (B' \to B))$. Therefore, either $V \subseteq V'$ or $V' \subseteq V$. □

Definition 11. *A subset Q of T is* sibling disjoint *if for each $V, V' \in Q$, $V \cap V' = \emptyset$. Consider the inclusion $\subseteq$ on the set of sibling disjoint subsets of T. A subset Q of T is* suitable *iff it is a $\subseteq$-maximal sibling disjoint. For each suitable $Q \subseteq T$, the* canonical partition *of Q is $Q \cup \{\{y\} : y \notin \bigcup Q\}$. A partition P is* canonical *if it is the canonical partition of some suitable Q.*

Lemma 12. *For each $V \in T$ there is a suitable subset Q of T such that $V \in Q$.*

Proof. Pick $V \in T$ arbitrarily. Enumerate T as $V = V_0, V_1, \cdots, V_n$, and let $Q_0 = \{V_0\}$. Define

$$Q_{i+1} = \begin{cases} Q_i \cup \{V_{i+1}\} & \text{if } V_{i+1} \cap \bigcup Q_i = \emptyset \\ Q_i & \text{otherwise} \end{cases}$$

Each Q_i is sibling disjoint and $V \in Q_n$. Obviously $Q_i \subseteq Q_{i+1}$. Therefore, Q_n is suitable and $V \in Q_n$ as desired. □

Lemma 13. *Let Q be a suitable subset Q of T, let the canonical partition P of Q. Then, $V \in Q$ iff there is B such that $V = \{C : C \vdash B\} \in P$.*

Proof. It is immediate from the definition that there is B such that $V = \{C : C \vdash B\} \in P$ if $V \in T$. Conversely, if there were any V such that there is B such that $V = \{C : C \vdash B\} \in P$, $V \in Q$ since otherwise Q cannot be suitable. □

For each $V \in T$ and each partition P of U_0, let $S(P, V) = \{U \in P : U \cap V \neq \emptyset\}$.

Lemma 14. *Let P be a canonical partition on U_0, $V \in T$, and $U \in S(P, V)$. Suppose $U \subseteq V$. Then $\bigcup S(P, V) = V$.*

Proof. Let us show that for an arbitrary $U \in P$, $U' \in S(P, V)$ iff $U' \subseteq V$. It is easy to see that $U' \subseteq V$ implies $U' \in S(P, V)$. For the converse, suppose $U' \nsubseteq V$. Then there is $x \in U' \setminus V$, which implies $U' \in T$. Then $V \subseteq U'$ must hold, while at the same time there is

$U \in S(P, V)$ such that $U \subseteq V$. It follows that $U \subseteq U'$, hence P cannot be a partition, against the assumption. Therefore, for each $U' \in P$, $U' \in S(P, V)$ iff $U' \subseteq V$. It implies that $\bigcup S(P, V) = V$. □

Lemma 15. *Let P be a canonical partition on U_0, $V \in T$, and $U \in S(P, V)$. Suppose $V \subseteq U$. Then $S(P, V) = \{U\}$.*

Proof. Suppose $U' \in S(P, V)$. Then $U' \not\subseteq V$ must hold, as otherwise $U' \subseteq U$ must hold, which implies P cannot be a partition. Hence, there is $x \in U' \setminus V$, thus U' is not a singleton, so $U' \in T$. Therefore, $V \subseteq U'$ must hold. Moreover, $U = U'$, as P is a partition. Therefore, $S(P, V)$ is a singleton. □

Proposition 16. *Let P be a canonical partition on U_0 and $V \in T$. Suppose there is $U \in S(P, V)$ such that $U \subseteq V$. Then there is a canonical partition P' such that $P \leq P'$.*

Proof. By Lemma 14, $\bigcup S(P, V) = V$ holds. It implies that $P' = (P \setminus S(P, V)) \cup \{V\}$ is a partition. Pick a suitable $Q \subseteq P$. Define $Q' = (Q \setminus \backslash S(P, V)) \cup \{V\} \subseteq T$ is also suitable, since otherwise Q cannot be suitable. Let P' be the canonical partition of Q'. Consider an arbitrary $U \in P$. If $U \in (P \setminus S(P, V))$, $U \in P'$. If $U \in S(P, V)$, $U \subseteq V \in P'$. Therefore, $P \leq P'$. □

Proposition 17. *Let P be a canonical partition on U_0 and $V \in T$. Suppose there is $U \in S(P, V)$ such that $V \subseteq U$. Then there is a is a canonical partition $P' \leq P$.*

Proof. By Lemma 15. $S(P, V)$ is a singleton $\{U\}$. Let $Q \subseteq P$ be a suitable partition. Define $Q' = (Q \setminus S(P, V)) \cup \{V\}$. Then, Q' is suitable, for if not, Q cannot be suitable either. Let P' be the canonical paritition of Q'.

Consider an arbitrary $U' \in P'$. If $U' \in S(P, V)$, $U' \subseteq U \in P$; if $U' \notin S(P, V)$, $U' \in P$. Therefore, $P' \leq P$. □

Proposition 18. *Let P, P' be canonical partitions on U_0 such that $P \leq P'$, and $V \in T$. Suppose that there is $U \in S(P, V)$ such that $V \subseteq U$ and that there is $U' \in S(P', V)$ such that $V \subseteq U'$. Then there is a canonical partition P'' such that $P \leq P'' \leq P'$.*

Proof. By Lemmas 14 and 15, $\bigcup S(P, V) = V$ and $S(P', V)$ is a singleton $\{U'\}$.

Define $Q'' = Q' \setminus S(P', V) \cup \{V\} \cup Q \setminus S_(P, V)$. Then $Q'' \subseteq T$ and Q'' is suitable. Let P'' be the canonical partition of Q''.

$P \leq P''$, by considering an arbitrary $U \in P$. If $U \in (P \setminus S(P, V))$, $U \in P''$. If $U \in S(P, V)$, $U \subseteq V \in P''$. Therefore, $P \leq P''$. Also $P'' \leq$

P'. by considering $U'' \in S(P', V)$. $U'' \subseteq U \in P'$; if $U'' \notin S(P', V)$, $U'' \in P'$. Therefore, $P'' \leq P'$. □

Lemma 19. *There is a $\leq$-chain Π of canonical partitions.*

Proof. Pick $V \in T$ arbitrarily. Enumerate T as $V = V_0, V_1, \cdots, V_n$. Fix an arbitrary suitable subset Q_0 of U_0 such that $V \in Q_0$, and let P_0 be the canonical partition of Q_0. Let $\Pi_0 = \{P_0\}$.

A chain of sets of canonical partitions is to be defined step by step. The construction in $k+1$'s step is as follows. Suppose Π_k is defined and its members are $\leq$-ordered as $P_{k_0} \leq \cdots \leq P_{k_k}$. V_{k+1} is to be considered. Notice that, since every set V_i in T is non-empty, in each partition there must be a member overlapping with V_i.

There are three cases.

(1) For each l ($0 \leq l \leq k$) and for each $V \in S(P_{k_l})$, $V \subsetneq V_{k+1}$.
Define $P_{k+1} = (P_k \setminus S(P_k, V_{k+1})) \cup \{V_{k+1}\}$. By Proposition 16, P_{k+1} is a canonical partition and $P_{k_k} \leq P_{k+1}$.
Let $\Pi_{k+1} = \Pi_k \cup \{P_{k+1}\}$ and extend the order $P_{k_0} \leq \cdots \leq P_{k_k} \leq P_{k+1}$.

(2) For each l ($0 \leq l \leq k$) and for each $V \in S(P_{k_l})$, $V_{k+1} \subsetneq V$. $S(P_k)$ must be the singleton $\{V\}$. Define $P_{k+1} = (P_k \setminus S(P_k)) \cup \{V_k\} \cup \{x : x \in \bigcup S(P_k) \setminus V_k\}$. By Proposition 17, P_{k+1} is a canonical partition and $P_{k+1} \leq P_{k_0}$.
Let $\Pi_{k+1} = \Pi_k \cup \{P_{k+1}\}$ and extend the order $P_{k+1} \leq P_{k_0} \leq \cdots \leq P_{k_k}$.

(3) There is l ($0 \leq l \leq k$) such that there is $V \in P_{k_l}$ such that $V \subseteq V_{k+1}$ and there is m ($0 \leq m \leq k$) such that there is $V' \in P_{k_m}$ such that $V \subseteq V_{k+1}$. Let P^l be the maximal among those such that there is $V \in P_{k_l}$ such that $V \subseteq V_{k+1}$ and P^m be the minimal among those such that there is $V' \in P_{k_m}$ such that $V \subseteq V_{k+1}$.
$P_{k+1} = P^m \setminus S(P^m) \cup \{V_{k+1}\} \cup P^l \setminus S_(P_l)$ By Proposition 18, P_{k+1} is a canonical partition. Moreover, $P_{K_i} = P^l \leq P^m = P_{K_{i+1}}$, and $P^l \leq P_{k+1} \leq P^m$.
Let $\Pi_{k+1} = \Pi_k \cup \{P_{k+1}\}$ and extend the order $P_{k_0} \leq \cdots \leq P^l \leq P_{k+1} \leq P^m \leq \cdots \leq P_{k_k}$.

Repeat the construction up to $k = n-1$, and let $\Pi = \Pi_n$. Obviously, Π_n is a set of partitions linearly ordered with respect to $\leq$.

□

Let $v : Prop \to \wp U_0$ as $v(p) = \{C : C \vdash p\}$. Define $M_l = \langle U_0, \Pi, v \rangle$.

Lemma 20. *$M_l, C \models B$ iff $C \vdash_{\mathcal{S}} B$.*

Proof. The cases where the outermost operator is not ∇ are similar to Proposition 6.

Suppose $B = \nabla B_0$. First, suppose $C \vdash \nabla B_0$. Then $\langle\exists\rangle\nabla B_0 \in X_A$, so $B_0 \in \Omega$, thus $X = \{C : C \vdash B_0\} \in T$. Thus, by Proposition 19, there is $P \in \Pi$ such that $X \in P$. Therefore, $M_l, C \models B$.

Conversely, suppose $M_l, C \models \nabla B_0$. It is equivalent to that there is $P \in \Pi_l$ such that $X \in P$, and $C \in X$. Since each P is canonical, $C \vdash \nabla B_0$. □

Corollary 21. *Suppose A is valid in the class of linear p-frame. Then $\vdash_{\mathcal{S}} A$.*

Proof. Given a $\mathcal{S}$-consistent A, there is $C \in M_l$ such that $M_l, C \models A$, since $A \in X_A$. Thus, there is a linear p-model of A. □

9. Further direction

An intended interpretation of a linearly ordered set of partitions is a situation with multiple contexts where a message bears information in various levels of details, as seen in everyday life where ambiguous messages often appear. Impossibility of characterization of the class of frames a linearly ordered partition set, while being formally interesting, looks despairing for the application.

It seems necessary to extend either semantics or language, or even both, to characterize such a notion. Nevertheless, it is still open whether a similar class of frames with a linearly ordered partition set is characterized in a system of any language.

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