MODAL LOGIC OF PARTITIONS

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Abstract
This thesis proposes “partitionistic elimination” in philosophical applications of modal logic based on possible world semantics to notions around information and to deontic reasoning.

There are two common ideas in usages of possible worlds in philosophical logic relevant to information and agency. First, elimination of possibilities represents a condition of information acquisition and real choice. It has been usually unspecified how many possibilities are to be eliminated. Second, a partition of a set of possible worlds often represents epistemic options or choices toward an action. The thesis proposes to combine these ideas: use a partition in order to specify how many possible worlds are to be eliminated when information acquisition or action arises.

Based on the proposal, the thesis develops semantics of a modal language whose logic is non-monotonic in general. An axiomatization of the basic logic of the semantics is shown. Completeness and finite model properties are proved. The thesis also characterizes some frame conditions, but a nested partition condition is not characterizable.

The formal results are applied to the two areas of information and obligation. The first application of new modal logics is to the logic of questions and answers. Modal operators formally describe logical behavior of relevant informal concepts previously argued. A formal counterpart of the notion of complete and just complete information behaves to elucidate some examples related to Gricean conversational implicature. The notion of partial information is to be refined to three subclasses with different logical properties. The second application of partition
-istic elimination is to paradoxes of deontic logic. In connection with the moral dilemma argument in ethical theory, the proposed logic behaves differently from Belnap’s see-to-it-that logic. The thesis argues that the proposed logic is more adequate as a continuation of the research project of deontic logic.

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CHAPTER 1

Introduction

1. Philosophical logic

Philosophical logic is a division of philosophy. It aims to clarify philosophical discussions via analyzing logical behavior of major concepts. Philosophical logic intends to analyze how concepts behave in philosophical discussions. It aims to give a sharp criterion of validity of arguments with the target concepts, and to magnify subtle details in arguments which might be overlooked with an intuitive grasp of the terms.

Formal and informal considerations complement each other to reach the goal. Since the twentieth century, formal methods have achieved a wider development than the traditional informal methods had in thousands of years. Medieval logic should be counted as an incredible quantity of projects in philosophical logic in this sense, for example, even with no mathematical treatments.

Conceptual analysis often invites formalization. Mathematical results about supposedly formal counterparts of the concept in question, then, are decoded to show consequences of their behavior and give feedback to philosophical arguments. Formal methods to philosophical discussions are as mathematics to theoretical physics: although philosophy had not heavily relied on mathematical theories until the twentieth century, and perhaps although most areas of philosophy still now develop with little help from formal logic, logical analysis has held an important place in the discipline of philosophy. Such a notion of ‘modern philosophical logic’ echoes examples of logical research in the latter half of the twentieth century, when philosophical applications of formal logic began.
A typical project in modern philosophical logic is equipped with five steps: conceptual analysis, formalization, system design, mathematical treatment, and interpretation of formal results. Not all projects of philosophical logic have all those five steps, however.

Conceptual analysis and formalization are interrelated steps. Conceptual analysis, on the one hand, typically begins with focusing on a concept in philosophical discussion, and looks for a way to paraphrase sentences where the target concept term appears into “a normal form” by extracting patterns of its behavior. Once a correct method to paraphrase is established, formalization should be easy; an adequate formal language for the whole project is available. Nevertheless, the formalization phase can be very difficult, as there might not exist any such formal languages in general. Design of any adequate formalism (or representation systems) must play a big role in such projects which initially have no appropriate formal language or representation.

Logical behaviors of the normal form, once extracted via conceptual analysis, are described at the next phase, i.e., the system design phase. Descriptions can be either syntactic or semantic, according to what motivates the analysis. The mainstream of philosophical logic seems to have the from-semantics-to-syntax direction. Investigations in the direction begin with a description of semantic structures to reflect insights and requirements from motivating philosophical discussions in the first phase of conceptual analysis. The primary goal of the from-semantics-to-syntax method is to determine which sentences are valid and what sort of formal arguments preserve validity. It is usually not so difficult to check the validity of each formula and valid argument. Such validity results are applied to the original philosophical arguments to yield finer logical analysis than their informal counterparts. Furthermore, it is meaningful to ask whether the logic, or the set of all valid formulas, can be axiomatized. For any formal system complete with the semantics surely will help to determine
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valid arguments. Axiomatizability is not in general guaranteed, however; in fact, some logics are shown not to be axiomatizable.

On the other hand, the reverse direction, the from-syntax-to-semantics direction, is adequate for some applications. In this direction, conceptual analysis and formalization determine what properties the concept term should bear. Then, the properties are syntactically described by additional rules and axioms. The main question in this direction often is to search for adequate semantic structure. Given a base logic (classical propositional logic or first-order logic, for example), deduction can be defined so that a set of theorems of the formal system will be calculated. It is desirable that corresponding semantic structures be found, but the existence of such structures is not in general trivial. The method of mathematical logic usually plays the main role in the system design phase. Investigations of properties of the formal counterparts form the third step. It is “logic” in a narrow sense. Such formal investigations of logical properties can be globally or locally focused. With a global focus, properties in question are common to many systems or structures. Results from other areas of mathematics often can be applied. On contrary, treatments with a local focus target only several systems or several sorts of structures. They are motivated by concrete applications. Because of conceptual analysis and formalization in the analysis, the systems and semantics may not have as nice properties as in global treatments, or they may look complicated with background analysis. They may even require ad hoc techniques to get desirable results, as standard techniques may not work.

Formal results obtained from a logical theory are applied, if successful, back to the original argument at the interpretation phase. It is required that the reinterpretation from formal results to informal arguments should preserve the formal properties. If correctly reinterpreted, formal results will make short-cut arguments concerning the
concept term in question; they also will give rigorous criteria of valid arguments in such discussions so as to check invalid arguments more easily.

So-called paradoxes in discussions of philosophical logic, or discrepancies between logical and intuitive properties, appear in the step of reinterpretation. Unlike liar-type paradoxes, these paradoxes are not really paradoxical, however. Rather, they just present counterintuitive phenomena, which stem from inadequate formulation. Applications of formal results to informal discussions may bring counterintuitive instances of arguments. The informal counterpart of a formally valid inference can appear invalid when it is invoked in front of our intuition. It can be the opposite: the informally valid argument may have a formally invalid counterpart argument. Such paradoxes, or discrepancies between the formal and the informal, suggest flaws in the first and second steps of the analysis before formal treatments. Nevertheless, they do not blight every philosophical application to the field of philosophical discussions; critical considerations of previously existing analyses may also motivate a new analysis.

2. Modal approach in philosophical logic

The modal approach is a branch of philosophical logic, where the language of modal logic is chosen in formalization. The syntactic approach dominated researches of emerging modern modal logic from 1900’s to early 1950’s.

The research aimed to find a logical system to reflect logical properties of implication in our everyday language, or English. It began with frustration against Whitehead and Russell’s material implication. Material implication is intended to formalize mathematical reasoning. When interpreted as implication or entailment in our everyday language, however, it does not behave well: everything is implied by a contradiction, and everything implies a logical truth. Such phenomena are called
paradoxes of material implication. C. I. Lewis proposed several systems of strict implication to solve such paradoxes by listing intuitively acceptable axioms\(^1\). Independence of those systems are first syntactically shown by deducibility, then semantically with matrices, or assignments of many-valued truth values. It may be seen as an attempt of searching for adequate semantics, but at that time it remained without completeness. It turned out that strict implication syntactically behave the same as combination of a modal operator and material implications\(^2\). Paradoxes of strict implication are again pointed out: everything is implied by the impossible, and everything implies the necessity. Ackermann proposed rigorous implication to avoid paradoxes of strict implication. The trend has been inherited to research of relevant logics.

The semantic approach in philosophical logic has developed since 1950’s, ever since the development of semantics of modal operators in terms of possible worlds, which gives philosophers the tool of formalization with the standard modal language and the standard Kripke semantics (Kripke\(^43\)).

A Kripke model is a set of points (possible worlds) and a binary relation (the accessibility relation), together with a truth-value assignment (a valuation) which tells whether or not an atomic proposition holds in each world. In such a structure, the truth condition of a modal operator mimics our naive understanding of the notions of necessity by taking each world as a possible world: a proposition is necessary if it is true in every possible world. Its formal counterpart is that a modal sentence \(\Box A\) in world \(w\) is true if and only if \(A\) is true in every world \(x\) that bears the relation \(Rwx\).

Given Kripke semantics, philosophical applications of modal logic along this line seem easy. Apparently, a project easily conforms to the typical procedure of philosophical logic: conceptual analysis, formalization, system design, technical treatment,
and feedback. The following exemplifies a wishful thought of an application of modal logic. In the conceptual analysis phase of a modal analysis, a sentence with the target concept is paraphrased into a combination of a sentential modifier and an assertive sentence. A sentential modifier in turn is formalized as a modal operator in the formalization phase. As such a modifier takes an assertive sentence as its argument, those paraphrased sentences can have well-formed formulas in a modal language recursively defined in the formal counterpart. The system design is already done. Thanks to this fact, techniques and results in modal logic give properties of the logic in question, so that the properties are reinterpreted in the motivating philosophical discussions.

In actual applications, the conceptual analysis may or may not suggest such straightforward paraphrases. Propositional attitudes are taken as sentential operators as they stand, for example, so that it is argued that they admit a treatment via the modal approach. Epistemic logic and doxastic logic aim to represent belief and knowledge as modalities, for example. Possible worlds are taken as epistemic and doxastic alternatives in Hintikka, whom C. I. Lewis might have influenced with his idea in the 1920s of “imaginabilities” or “conceivabilities” as the notion of possible worlds. Giving semantic conditions of the modalities constitutes the core of formalization in the case of epistemic logic, for example. Possible worlds in a formal structure are supposed to simulate epistemic/ doxastic alternatives, and the accessibility relation is to determine which epistemic/ doxastic alternatives are possible in each world. The supposition that we reflect on our knowledge; that is, that we have knowledge of our knowledge is reflected in transitivity of the accessibility relation.

\footnote{C. I. Lewis’s position is much closer to what Stalnaker later calls actualism. He characterizes: “A fact is what makes some true proposition true: for every true proposition there is some corresponding fact, and every fact is expressible by some proposition which is true [45, p.85].” A similar characterization also appears in [44, p.384]. He furthermore claims that facts are not objects, and even steps forward to allow \textit{unreal} facts in addition to actual facts.}
The supposition that knowledge of a proposition implies that the proposition is true makes the accessibility reflexive.

Concepts around auxiliary verbs are also considered to fit the modal approach. Standard deontic logic takes permission and obligation as sentential operators. In deontic logic, possible worlds are classified into the morally ideal worlds and the others, and obligation of a proposition \( A \) is formulated as \( A \) holds in every morally ideal world. The accessibility relation connects each world to every morally ideal world; the philosophical assumption that there must exist at least one morally ideal world turns out to correspond to the logical property of deontic notions that obligation implies permission; i.e., if \( A \) is obliged, then the \( A \) must be possible.

Temporal logic incorporates modal operators which are intended to be read as past, present, and future, and also to formalize tensed sentences, which had been considered illegitimate subjects of formal logic. They were considered to be residents in the realm of pragmatics before Prior proposed the idea of temporal logic\(^4\).

Further applications of the modal approach require non-straightforward paraphrases. Hamblin’s modal analysis of imperatives paraphrases, for example, an imperative sentence “Stay here” into an assertive sentence “I request you to stay here.” Another example is Åqvist’s analysis of questions\(^5\). It argues for these paraphrases, following the tracks of Hamblin’s logical analysis of imperatives. He is the first to explicitly formalize questions as a combined modality of imperative and epistemic modalities. According to his analysis, asking if \( A \) or not means the questioner does not know if \( A \) and she requests to erase the ignorance. It follows that an utterance of “is it \( A \)?” should be paraphrased as an utterance of “let me know if \( A \)”, a combination of imperative and knowledge. He also introduces the paraphrase from imperatives to

\(^4\)[51].
\(^5\)[1].
assertive sentences with an imperative modality. Those nonstraightforward applications requires such paraphrases to work, and the direction of conceptual analysis should be rather from syntax to semantics, to look for adequate semantics.

Nevertheless, those pioneering applications of modal logic oversimplified linguistic phenomena. “Paradoxes” were pointed out almost immediately after those initial applications. The generic form of paradoxes is that those modal operators do not reflect properties of the targets of analysis; our intuitive notions behave differently from their formal counterparts.

For example, the main paradox of epistemic logic, called *logical omniscience*, points out a formal property which our everyday notions of knowledge and belief do not bear: logical truth must be known and believed. Paradoxes of deontic logic, to be discussed later in Chapter 3, also show discrepancies between our intuitive notions around agency and their formal counterparts. Some valid inferences of standard deontic logic should be invalid in our practical reasoning.

While such paradoxes disappointed philosophers working on the modal approach so that they hesitated to apply the logical analysis of sentential modifiers to philosophical discussions, logicians did not give up the formal method; instead, they have tried to develop modal logic to accommodate our everyday usages of the words.

The observation leads to the following questions: Why have philosophical applications of modal logics failed? What is the source of oversimplification in the approach?

\footnote{While his conceptual analysis and paraphrase are insightful, the formal analysis does not work very well. Åqvist’s formalization takes four modalities: epistemic ($K$, $S4$), imperative ($!$, $K$), and their duals. Since the modalities are normal, the formalization suffers from some problems. First, the problem of logical omniscience applies. Second, $!Kp \rightarrow ip$ is a theorem, which means under the intended interpretation that if you ask $p$, then $p$ is morally permissible. It is counterintuitive and absurd. Since he formalizes his analysis with normal modalities, however, paradoxes are unavoidable.}

\footnote{Methodological criticism by Quine and his followers also discouraged philosophers to use modalities.}

\footnote{Another direction logicians have taken is applications to computer science, in particular to artificial intelligence. In fact, many logicians have immigrated to computer science. It turned out to become the mainstream of the research area now.}
The questions are too broad and need to be specified to be answerable. This thesis thus aims to spell out a question whose answer should give a partial answer to those big questions.

3. Previous approaches to paradoxes of modal logic

There were two starting points to solve paradoxes of modal logic. The first starts from the analysis that the paradoxes are caused by monotonicity of modal operators; i.e., the property that, if \( A \rightarrow B \) is a theorem, so is \( \Box A \rightarrow \Box B \). In epistemic logic, for example, with \( \Box \) read as “it is known that” or “it is believed that”, monotonicity yields logical omniscience. The argument is this. For an arbitrary \( A \), \( A \rightarrow \top \) is a theorem; as \( \Box A \rightarrow \Box \top \) follows, monotonic epistemic logic should tell, if anybody knows any proposition, the person should know any logical truth. That is absurd. Thus, monotonicity should not be a valid inference rule in a logical system whose intended interpretation is epistemic.

Others criticize material implication, which does not fit our reasoning involving those intensional issues. This approach leads to trials of formalizing non-monotonic implications as binary modal operators. Both analyses argue against monotonicity\(^9\); logicians have proposed non-monotonic modalities to represent the notions of philosophical interests, such as action, knowledge, and obligation.

It must be noticed first, however, that bare non-monotonicity does not solve the paradox. Non-monotonicity is a necessary condition of a nice modal logic for the applications. It is not a sufficient condition. In fact, as argued later in Chapter 3, new

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\(^9\)Non-monotonic reasoning in computer science is different from what is argued here. Non-monotonicity in non-monotonic reasoning means that adding an assumption does not preserve validity of inference. For example, even when the reasoning

\[
\text{John is a bird; therefore, John flies.}
\]

is valid, the following reasoning with an additional assumption

\[
\text{John is a bird; John is a penguin; therefore, John does not fly.}
\]

is also valid, although the consequence of the latter negates that of the former.
paradoxes can still be constructed with a non-monotonic logic. Of particular interests here is the see-to-it-that (STIT) theory of agency (Belnap et al. [6]). The STIT approach seems to provide nice solutions to paradoxes of deontic logic by proposing several non-monotonic modal logics. Nevertheless, this thesis will step forward to claim that more paradoxes which even STIT approaches cannot evade give evidence that mere non-monotonicity is not the real cause of paradoxes.

4. Proposal

My proposal below aims to give a philosophical basis for a semantics of new modal operators to evade paradoxes of modal logic, in particular, paradoxes of epistemic logic and deontic logic. It is to be examined in applications in later chapters. Let us begin with some observations in both epistemic logic and deontic logic.

4.1. Elimination of possibilities and information. The idea of elimination of possibilities has been observed in modal approaches in both applications.

Connection between the notion of possibility and information has been explicitly stated in 1925 by R. A. Fisher

Shannon’s definition of information furthermore related the notion of information with entropy. It follows that the message bears no information if it eliminates no state or all the states in the possibility space.

---


What we have spoken of as the intrinsic accuracy of an error curve may equally be conceived as the amount of information in a single observation belonging to such a distribution. If \( p \) is the probability of an observation falling into any one class, the amount of information in the sample is \( S\{(\partial m/\partial \theta)^2/m\} \) where \( m = np \), is the expectation in any one class [and \( \theta \) is the parameter].

Many authors who pursue qualitative notion of information also put the eliminative view of information in their own term.

Grice characterizes information by non-triviality in his first conversational maxim of Quantity: make your contribution to a conversation as informative as is required but no more. It follows that tautologies, which cannot be otherwise, are non-informative:

Extreme examples of a flouting of the first maxim of Quantity are provided by utterances of patent tautologies like *Women are women* and *War is war*. I would wish to maintain that at the level of what is said, in my favored sense, such remarks are totally noninformative and so, at that level, cannot but infringe the first maxim of Quantity in any conversational context[25, p.33].

Carnap and Bar-Hillel [2] attempt to characterize the notion of semantic information to identify it with the theory of inductive probability relative to L-truth. Nevertheless, it does not properly reflect our everyday notion of information which is context-dependent. The Carnapian L-truth should be relaxed to incorporate our intuition that information should depend on our background knowledge and the context.

Dretske [16, p.4] provides the eliminative view in (quantative) information theory:

Information theory identifies the amount of information associated with, or generated by, the occurrence of an event (or the realization of a state of affairs) with the reduction in uncertainty, the elimination of possibilities, represented by that event or state of affairs.

He characterizes his notion of *information content* as:
1. INTRODUCTION

A signal $r$ carries the information that $s$ is $F$ iff the conditional probability of $s$’s being $F$, given $r$ (and $k$), is 1 (but, given $k$ alone, less than 1) \[16, \text{p. 65}\]

Stalnaker also describes relationships between information and possible worlds\[63, \text{p.6}\]:

An assertion can then be understood as a proposal to alter the context by adding the information that is the content of the assertion to the body of information that defines the context, or equivalently, by eliminating from the context set—the set of possible worlds available for speakers to distinguish between—those possible worlds in which the proposition expressed in the assertion is false.

From this line of thoughts, Barwise [4] extracts the inverse principle between possible worlds and information: the more information is, the less possibilities are; and vice versa.

Thus, elimination of possibilities is identified with obtaining information in theories of qualitative information as well as in the communication theory, i.e., the theory of quantitative information.

4.2. Elimination of possibilities and agency. A modal approach to the notion of agency can incorporate the idea of elimination of possibility in the form of “principle of no laissez-faire”: a real choice of an agent is represented as interference in the situation\[12\]. Interference in the situation is in turn represented as elimination of possibilities as von Wright, von Kutschera, and Pörn support the idea among other authors. Thus, the key idea is that a voluntary action to do $A$ requires that non-$A$ could happen without the action in question, while any action of an agent should be

\[12\text{Chellas [11]}\] also works on the notion of agency. Nevertheless, his approach is different, and does not have the idea of elimination of possibilities.
based on the agent’s choice (Counterfactual condition). In other words, A cannot be voluntarily chosen, however, when A necessarily happens.

Modal logics of agency thus can be regarded as a class of logics with operators with truth conditions involving the minimal negative condition for agency:

There is a possible world to be eliminated as the undesirable holds in the world (negative condition)

4.2.1. Elgesem’s minimal logic. Minimal logic of agency (Dag Elgesem) [17] implements the idea by function semantics. Governatori-Rotolo [24] axiomatize the logic, and restate the semantics in possible worlds.

4.2.2. See-to-it-that theory. The see-to-it-that theory of agency [6] has more semantic requisites than Elgesem’s. It is supposed to reflect the notion of real choices, so that the game theoretical notion of strategy is introduced on branching time semantics. Details will come in Chapter 3.

4.3. Proposal. Though the idea of elimination of possibilities appears in various aspects in philosophical logic using possible worlds, how many possible worlds are to be eliminated is not much argued. It seems natural to ask the question: how many possibilities should be eliminated?

The most popular way to go, it seems, is probability. There are two approaches: frequentist and Bayesian. Frequentism takes probability as relative frequency. On the other hand, Bayesianism considers probability as degree of confirmation. The main objection against Bayesian probability points out that such a degree has no objective basis. Both approaches are quantative to assume a probability measure on possibilities

13 Carnap [10], while comparing the two approach, points out that the frequentist approach does not fit inductive reasoning. He also argues the classical notion of probability, which is defined as the ratio of the number of favorable cases to the number of all possible cases [10, pp.23-33].
Nevertheless, the assumption further requires to clarify ontological status of possible worlds. Is there any other way to control underlying the idea of elimination, with no commitment to ontological status of possible worlds? A key to a qualitative measure of calibres seems to be found in usage of partitions.

This thesis rather proposes to use partitions of a set of possible worlds. The idea is almost ubiquitous. *Partitionism* is the idea that reason functions since it classifies things. The idea comes into modal approach when it takes the form that reason classifies possibilities.

In various aspects of our life, the idea emerges that human reason functions on basis of classifying the perceived or the captured into a partition. Black-and-white points on a display, for example, may indicate a sign of disease to trained professionals while they form mere a meaningless pattern to an average patient; experimental data turn out to be a significant result only if they are adequately interpreted under an appropriate theory in a particular situation; and decisions are made among available choices by comparing possible cases in a particular situation, which are mutually exclusive but exhaustive.

Such a way of thought, partitionism, can take various forms and levels according to the subject of classification process. There are roughly two sorts: perceptual partitionism and judgmental partitionism. In perceptual partitionism, reason classifies data. Without such a classification, raw data would make little sense by themselves, as Kant’s saying: Thoughts without content are empty, intuitions without concepts are blind\(^\text{14}\). Segments arise through the classification process, and then reason articulates them to prepare for further judgment.

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On the other hand, in judgmental partitionism reason classifies possibilities or cases. Reason functions in choosing from cases in deciding what to do or in obtaining information from investigation or segments which form themselves through different levels of classification process. Judgmental partitionism often accompanies modal realism, which concerns on metaphysical status of possible worlds and says nothing about partitions by itself. Appealing to modal realism, each case in a particular situation can be represented by a collection of possibilities. Analysis of conditional inferences often commits itself to the version of judgemental partitionism, for example.

The paper mainly focuses on judgmental partitionism, on which modal logics on partitions are to be proposed to apply to philosophical discussions. It can be classified into two subcategories according to which position of modal realism it incorporates. First, epistemic partitionism aims to describe how information is acquired by means of possible worlds. Inspired by Shannon, many authors who work on qualitative aspects of information have advocated the position. On the other hand, causal partitionism reflects metaphysical modal realism. The formal similarities between them allow unified investigations into logics of partitions, while the notions of possible worlds in those two approach differ from each other.

Thus, my main proposal is simple. Two ideas in philosophical logic are to be combined: elimination of possibilities and partitions of a possible world set.

Proposal (Partitionistic elimination)

Use a partition to decide how many possibilities to eliminate. Elimination of possible worlds should be relativized to available partitions. Elimination of possibilities is significant in a context (represented as a set of partitions of possibilities) when it leaves only one piece in a non-trivial partition in the context.
1. INTRODUCTION

Technical issues should be neutral in philosophical discussions. It is just a matter of what application is intended, however. In fact, a nice subeffect of the method of partitionism is that it allows to define modalities with partitions. Applications in the following chapters of the thesis are based on properties of such partitionistic modal logics argued in the technical part. As ubiquity of partitionism may suggest, partitionistic modal logics seems to be applicable to various areas.

5. Summary

5.1. Chapter summary. There are two common ideas in usages of possible worlds in philosophical logic concerning information and agency: elimination of possibilities and application of partitions of possibilities. In dealing the notion of information, one of the main target in such applications, many authors advocate the inverse principle between information and possibilities: the more possibilities are eliminated, the more information is obtained; and vice versa.

The notion of choice is another main target. A real choice for agency is identified with no laissez-faire. In terms of possible worlds, the bottom line condition for agency is to eliminate undesirable possibilities. The upper limit of elimination is given by the condition of making a real choice.

Partitions of a set of possibilities are the other common idea to represent options. When information is concerned, on the one hand, a partition gives a set of answers in the context. On the other hand, notions about agency can be theorized with a partition as a representation of a set of choices in the context.

My proposal is to combine those two ideas. Use partitions as measures of elimination of possibilities in both applications. We will examine in the later chapters that modal operators defined with partitions of a set of possible worlds reflects logical properties of target concepts in both logic of questions and answers and deontic logic.
5.2. **Outline of the thesis.** The main result of the thesis is argued in Chapter 6, which is devoted to modal logics arising from partitionistic structures.

The first application is logic of questions and answers in Chapter 2. Partitionistic elimination coincides with the previously proposed notion of complete and just complete answer in the study of erotetic logic. Such an answer bear just enough information for the inquiry. Logic of partial information is also argued in Chapter 7.

The second application concerns deontic logic. Chapter 3 will discuss an application of partitionistic elimination to paradox of deontic logic. While the stit theory solves paradoxes of deontic logic, addition of an intuitively natural assumption brings a strengthened version of Ross’ paradox. The main proposal in the thesis, partitionistic elimination, solves the stronger version.

Chapters from 5 to 7 are dedicated to technical backgrounds. Chapter 5 summarizes basic definitions and technique in modal logic: possible world semantics, neighborhood semantics, completeness, decidability among others. Chapter 6 discusses modal logics on partitions. It is intended to model complete and just complete answer in Chapter 2 and imperatives in a strict sense in Chapter 3. Chapter 7 gives technical backgrounds to the proposal of classifying the notion of partial answer in Chapter 2.
CHAPTER 2

The logic of questions and answers

1. Motivating examples

1.1. Twenty questions. Twenty questions is a two-player game to find what the answerer has in mind by posing at most twenty yes-no questions. In this example, suppose that the range of possible answers is a set of numbers from 1 to 1,000,000. Since a million is a bit less than \(2^{20}\), a wise questioner must be able to find the correct answer. How can she reach it?

Questioners may or may not be wise. For example, a questioner may ask “Can it fly?” to lose a precious opportunity; the question could yield no meaningful answer. Another bad way is to ask “Is the number less than 500,000?” in the first turn and get the answer yes, and then to ask “Is it more than 750,000?” Obviously, the answer is no, and he will lose a question without obtaining any information.

Another poor way is to repeat a question. Suppose the question “Is the number less than 500,000?” in the first turn gets the answer yes. Repeating the same question “Is the number less than 500,000?” would obtain the answer yes, but it does not help the questioner to squeeze the range of possibilities any more.

Now, it is natural for logicians to ask the following questions. What logical properties do such poor ways of asking questions bear? Can they be characterized?

Let us return to focus on a wise questioner. A systematic strategy is to cut the range in the half in each round. For example, asking “Is the number less than 500,001?” in the first turn submits a binary partition of the original range: from 1 to 500,000 for the one hand and from 500,001 to 1,000,000 for the other hand. The
question will get an answer “yes”, if the correct answer is 370,247. Then, in the second round, she could ask, “Is the number less than 250,001?” and will get an answer “no”. Then, she could continue, “Is the number between 250,001 and 375,000?” to obtain the answer “no,” and so on.

In such a strategy, each yes-no question in the sequence poses a binary partition of the range of possible answers. Since the range is cut in a half in each round, the question at the next stage will cut the remaining in the half again. Thus, a series of questions following the systematic strategy can be represented as nesting partitions.

We can ask ourselves again: is there any characterization of systematic strategies for this game? In particular, can any modality capture their logical properties?

1.2. Muddy children. The muddy children puzzle is another game where questions and answers play the central role. Let us consider the case of three players, a, b, and c. Every player has a playing card and can see the card of all the players but herself. An observer comes and says, “at least one of you has a red card,” and then asks: “do you have a red card?” Suppose that everybody has a red card. Each player sees the others’, and says “I don’t know” in the first round. The same happens in the next round. Suddenly in the third round, everybody says, “Yes, I have a red card.” What information is obtained by the iterated utterances of “I don’t know”?
The standard analysis of the muddy children puzzle with possible worlds also casts a light on the use of possibilities in information. The situation with the announcement “at least one of you has a red card” is described using possible worlds as follows:

In the diagram, each node represents a possible world, and each line with a label $a$, $b$, or $c$ represents an indiscernibility relation. $a$ cannot distinguish $abc$ and $\bar{a}bc$, for example, because $a$ cannot see her own card.

The possibility that all players have a non-red card (represented by $abc$ in the diagram) is eliminated by the initial announcement. Let nodes in parentheses represent eliminated possibilities. The indiscernibility relations connecting an eliminated possibility to others also disappear. With the possibility which is incompatible with the announcement being eliminated, any player should reason that she herself must have a red card, if she saw there is no other player who has a red card.

The answers “I don’t know” in the first round eliminate the possibilities that each player sees that no other players have a red card. The reason is that, if any player saw that there is no other player who has a red card, she should have already know that she herself must have a red card at the moment of the initial announcement. Therefore, the answers “I don’t know” imply that every player sees at least one other player who has a red card.
The apparently same answers “I don’t know” in the second round eliminate the possibilities in which two players have a red card. The resulting diagram has the only surviving possibility \( \overline{a\overline{b}\overline{c}} \), i.e., the possibility that every player has a red card.

The point of the puzzle is that the apparently-same answer “I don’t know” eliminates different possibilities at each round so that only one possibility survives after the final round. Are there any differences between logical properties of the answers in the final round and those beforehand? Can any modalities represent the differences?

2. The logic of questions and answers: a previous treatment

Logical investigations of questions and answers are not new. The area of logic of questions and answers has developed since the 1950s; a short history is summarized in Section 9 in this chapter. Informal analyses in the research in the 1950s aim:

1. To clarify logical relations between a question and its answers;
2. To formulate logical relations (equivalence and implication, for example) among questions in a context; and
2. THE LOGIC OF QUESTIONS AND ANSWERS

(3) To classify answers to a question into subcategories according to their logical properties in a context, and characterize those subcategories.

In particular, Belnap-Steel [7] develop a logical theory of questions and answers. Their central task is twofold: to construct a formal language to describe the roles of questions and answers in linguistic communication on the one hand, and on the other hand to conceptually analyze erotetic notions. They first distinguish interrogatives as sentences in an everyday language from questions as abstract objects\(^1\).

In their analysis, a question is represented as a combination of a presupposition and a set of mutually incompatible and exhaustive answers. Each of the mutually incompatible and exhaustive answers is called a *direct answer*. A direct answer to a question is a statement to answer the question completely and just completely, i.e., gives information just enough to meet the questioner’s expectation. A direct answer leaves no further inquiry for the questioner.

It is not assumed that every question has at least one true answer, although it might be expected to hold. Instead, a question is called *foolish* when it has no possible direct answer. A question is *meaningful* when it is not foolish. A question is called *trivial* when it has only one possible direct answer, or there is no alternative answer. Answers to a question are sorted out with respect to its direct answers.

- An answer is *complete* if its truth set is non-empty and it is a subset of a direct answer.
- An answer is *partial* if its truth set is a superset of a direct answer.
- An answer is *foolish* if its truth set is disjoint from the presupposition set of the question\(^2\).

\(^1\)The distinction is analogous to declaratives and propositions, multi-multi correspondences exist between them.

\(^2\)Thus, a foolish question must have a foolish answer.
• An answer is complete and just complete if its truth set is a direct answer.
• An answer is informative if its truth set is a proper subset of the presupposition set.

A similar set of notions can be introduced to speech acts of a wider range by taking a request as a set of direct responses.

2.1. Twenty questions with logic of questions and answers. Now we have an analyzing device in hand.

2.1.1. A foolish question. The question “Can it fly?” in the initial example is foolish. It could yield no meaningful answer.

Another foolish way is to ask “Is the number less than 500,000?” in the first turn and get the answer yes, and then to ask “Is it more than 750,000?” The second question is foolish, as the range of possible answers does not contain the correct answer at all, i.e., all possible answers to the question are foolish.

2.1.2. A trivial question. Repeating a question is another poor way, because repeated questions turn out to be merely trivial. Their true answer does not eliminate any possibilities, i.e., it is not an informative answer.

2.1.3. Complete and partial answers. An ideal strategy of the questioner is to systematically narrow down the range of possibilities. On the other hand, the answerer may keep the range of possibilities as large as allowed by the rules of the game. Thus, the answerer is considered to be poor if he gives the following sorts of answers to the question “Is the number less than 500,000?”

• “The number is between 1 and 700,000” or “The number is between 300,000 and 600,000.” Both give neither “yes” nor “no” to the given question, and he violates the rules of the game. Both answers are informative. The former is a partial answer, while the latter is not, to the given question.
• “The number is between 300,000 and 400,000” is a complete answer to the given question. The answer is indeed true and informative, if the correct answer is 370,247. It spoils the game, however.

The relationship among the notions of complete, partial, and complete and just complete answers can be illustrated in the following picture. Those notions are dependent on the receiver’s expectation, which is represented by a partition $P$. A partial answer is a superset of a member of $P$, and a complete answer is a subset of a member of $P$. A complete and just complete answer coincides with a member of $P$.

\[ P \]
\[ \text{complete} \]
\[ \text{partial} \]
\[ \text{CJC} \]

2.2. **Muddy children with logic of questions and answers.** The same conceptual device can describe the difference between the answers in the final round and the other answers beforehand. Only the former is complete and just complete to obtain the intended answer “Yes, I have a red card.” The answers in the latter case are merely partial answers. They are not sufficient to specify either of the intended answers, “yes” or “no”.
3. Multiple contexts with questions

Questions often can be answered in various ways. A single interrogative sentence may pose more than one question at the same time.

**Example 1.** Twenty questions. The answerer can give a true and complete answer if the questioner gives a meaningful question. It is the best legitimate way for him to give just “yes” or “no”, either of which is a complete and just complete answer.

**Example 2.** Muddy children. The interrogative “Do you have a red card?” can be answered at least by the two different sets of legitimate answers. The first set consists of the answers “Yes, I have a red card” and “No, I do not have a red card.” The second set consists of the answers “I know that I have a red card,” “I know that I do not have a red card,” and “I do not know whether I have a red card.”

The setting of the muddy children puzzle entertains the ambiguity in addition to veridicality of epistemic modalities. It does not count the first set as a legitimate set of answers in the first several rounds.

Obviously, asking a question for information is not straightforward. It depends on contexts, including surrounding situations, the previous knowledge, and the answerer’s knowledge of the questioner’s knowledge. Compare the following two examples for contrast.

**Example 3.** Muddy children with background knowledge on the colors of the suits. Suppose the players have background knowledge about standard sets of playing cards. If somebody sneaks in the room and declares “all of you have a diamond,” every player can answer “yes” to “Do you have a red card?”

**Example 4.** Muddy children without background knowledge on the colors of the suits. Suppose there is a player who does not know that diamond is a red suit. He
cannot conclude that he has a red card even with the announcement “all of you have a diamond.”

Economy of communication matters, too. In our everyday life, it is considered to be desirable to answer to a given question according to the expectation and background knowledge of the questioner.

Example 5. Asking where the Sample Gates are in Bloomington in August bears the information that the questioner is new on Indiana University campus as well as that she needs to know where the gates are. Utilizing the information, a wise answerer tries to answer the question mentioning nothing that the only old residents should know. An answer “The Sample Gates are at Kirkwood and Indiana” would bring little information for those who have no idea where Kirkwood is.

Nevertheless, the same answer would be trivial for old residents, too, if they pose the same question. They should know it already. What they might expect is, for example, the exact street address of the gates.

Example 6. “What treatment is needed for a patient of typhoid?” Suffering patients would be satisfied with answers such as “Antibiotic shots will work in addition to hospitalization for fluid therapy,” and would not care much about more information such as the kind of antibiotics and the recommended amount of the shot. Rather than listening to a long answer, they want to go to a hospital as soon as possible. Even a simpler answer “hospitalization” suffices.

Medical providers, however, would need more detailed information to save the life.

Example 7. There are several ways to answer, when I am asked “Where are you from?” To those who I meet in Bloomington for the first time, I would say “From Japan.” Usually they continue to ask “Which city?” I add “From Tokyo.”
To my new friends in Tokyo, however, I would answer them with local information. I may mention railroad stations near my home. If I answered “From Japan” to them, they would be perplexed. They already know that I am from Japan, and see no point in my saying “From Japan.” The answer is trivial, and they will suspect that there might be hidden assumptions or an unexpected intention.

Such local information will not work on Hoosiers. They cannot get any idea where I am actually from. When the receiver is not able to figure out in which country the station or the landmark I give is, the set of possible worlds in his view should be classified so that, for each country, there are possibilities that the station exists there.

Thus, it seems natural to assume that there are many layers of legitimate answers to a single question, in general. Some layers work, and the others do not. It depends on contexts.
The presentation of the logic of questions and answers in the last section focuses on only a single layer of answers. Belnap and Steel take such multiple layers of questions into consideration. They conceptually separate interrogative sentences and questions so that a single interrogative in an everyday language can bear multiple questions. To model situations involving questions and answers in an everyday language by possible worlds, it seems better to prepare with multiple layers of possible answers.

4. Proposal

Can the modal approach do anything in logic of questions and answers? Yes.

4.1. Proposal. My proposal for the logic of questions and answers is this. The main idea of this thesis, partitionistic elimination, is that complete and just complete answers should be taken as the only legitimate answers in each occasion. Focusing on complete answers leads to variants of logical omniscience; to avoid it, we need specification of a legitimate set of complete and just complete answers. It can be semantically formalized with multiple partitions, and the resulting modal logics are non-monotonic so that they avoid logical omniscience.

4.2. Possible world formalization. First, possible world semantics can be used to formulate the concepts informally described in the last section. A partition corresponds to a set of direct answers, or a question. A set of partitions describes the set of acceptable ways of answers.

Second, each notion of the logic of questions and answers has a corresponding modal operator as its formal counterpart.

4.2.1. Complete and just complete answers. In applications to the logic of questions and answers, the unary modality $\nabla$, argued in Chapter 6, is interpreted as
the indicator of complete and just complete answers\(^3\). Its intuitive reading is this: when \(\Box A\) holds at world \(w\) in model \(M\), a message \(A\) serves as a complete and just complete answer according to the receiver’s expectations. The logic of \(\Box\) calculates logical relationships among sentences which meet the receiver’s expectations in the situation of such a message transmission.

**Example 8.** Let us return to the example “Where are you from?” when the hearer is a person who meets me for the first time in Indiana. The formal language for the situation has atomic propositions \(\text{Japan}, \text{Taiwan}, \text{SouthKorea}, \text{Tokyo}, \text{Seoul}, \text{Taipei}\) among other propositions involving countries and cities she knows\(^4\): \(\text{Japan}\) stands for “I am from Japan”, for example. Each possible world represents a combination of a country and a city. Since the language here represents the situation that she knows the relationship of each city and each country, there are only combinations reflecting the real situation; i.e., \{\text{Seoul}, \text{Korea}\}, \{\text{Taipei}, \text{Taiwan}\}, etc. One of which is the actual world that represents where I am from. Assuming that one’s home country uniquely exists, we can pick up a partition of the set of possible worlds. She is furnished with two levels of answers, the country level and the city level. The city level is a subpartition of the country level. In such a situation, she will get complete and just complete information from my answer “From Tokyo”.

Nevertheless, she would be perplexed by my answer “Ichigaya” if she does not know where it is. Adding \(\text{Ichigaya}\) to the language splits possible worlds: then in the new model, the set of epistemic/doxastic possible worlds includes \{\text{Ichigaya}, \text{Tokyo}, \text{Japan}\}, \{\text{Ichigaya}, \text{Seoul}, \text{Korea}\}, \{\text{Ichigaya}, \text{Taipei}, \text{Taiwan}\}, and so on. The latter two do not reflect the real world, as Ichigaya is in fact the name of a station in Tokyo.

\(^3\)\(\Box\) allows several readings. An imperative reading is argued in Chapter 3.

\(^4\)She may not know some cities in those countries. Adding their names would yields the same situation as one with additional local information with no connection to other levels.
Unless she is well-prepared with the level of answers, she would not get complete and just complete information from such a message. Moreover, even if she guesses that Ichigaya is some area in a city, she will not be able to infer that I am from Tokyo, unless she does know in which city it is. This situation is represented by partitions which are not nested in each other.

**Example 9.** Negative question. Some linguists oppose the partitionistic approach by pointing out that those two interrogative sentences are taken as a same question in every context in the partitionistic approach.

- Is a whale mammal?
- Isn’t a whale mammal?

It is not a problem. We should consider that multi-multi relation between interrogatives and questions as well as declaratives and propositions. Then, we can say that those interrogative sentences represent the same question.

A possible objection is that, if equivalent, differences of usage between two interrogatives should be reflected as failure of logical equivalence between them. The answer to such a question is, however, that it is not necessary. Logical equivalence among questions should be not supposed to reflect pragmatic aspects of our usage of questions. Even for declaratives, the notion of logical equivalence is much coarser than the identity of meaning. Consider mathematical statements. The sentence that $1+1=2$ does not mean the same as the sentence that Fermat’s theorem holds.

**Example 10.** Negative answer.

- Who passed the exam?
  - John.
  - Not John.
The second answer may or may not make sense. Multi-partition settings can specify cases in which the receiver expects it. Consider the scenario: Mary just transferred to a new school. She became acquainted with John, but she has not talked with the other classmates yet.

In fact, negative answers are usually partial. If Mary could identify all her classmates, the answer “Not John” would be equivalent to the disjunction of all the classmates except John. The problem is that the question does not exactly specify the range of possible answers.

5. Modal logic on multi-partition

The possible world formulation easily gives truth conditions for information-related modalities: For example, where $\star A$ is read “$A$ is informative” (in the eliminative view), the truth condition will be $M, w \models \star A$ iff there is $w'$ such that $M, w' \nvDash A$. Another modal sentence $\star A$ whose intended reading is that $A$ is a complete and just complete message can be interpreted as: $M, w \models \star A$ iff for any $w', Rww'$ if and only if $M, w' \models A$.

The notion of informativeness in the logic of questions and answers is also implemented in terms of possible worlds: A message is informative with respect to a context if all possible worlds in an element of the partition are compatible with it.

The notion of complete and partial are restated as:

- A message is complete with respect to a context if only possible worlds in an element of the partition are compatible with it.
- A message is partial with respect to a context if all possible worlds in an element of the partition are compatible with it. (So the modality ‘··· bears partial information’ should behave as S5).
Also the notion of complete and just complete information: A message is just informative with respect to a context if all and only possible worlds in an element of the partition are compatible with it.

5.1. Semantics and formal results. See Chapter 6 for details of formal definitions and results.

A p-frame is \( \langle W, \mathcal{P} \rangle \), where \( W \) is a non-empty set, and \( \mathcal{P} \) is a set of partitions on \( W \), i.e., each \( P \in \mathcal{P} \) is a partition of \( W \). A valuation \( v \) on p-frame \( F = \langle W, \mathcal{P} \rangle \) is a function which assigns a subset of \( W \) to a propositional letter. A p-model on a p-frame \( F = \langle W, \mathcal{P}, v \rangle \) is \( \langle W, \mathcal{P}, v \rangle \), where \( v \) is a valuation on \( F \).

Let \([x]_P\) stand for the member \( U \in \mathcal{P} \) such that \( x \in U \).

The truth condition for \( \nabla \) is: \( M, x \models \nabla A \) iff there is \( P \in \mathcal{P} \) such that \([x]_P\) is the truth set of \( A \) in \( M \); and the truth condition for \([\forall]\) is: \( M, x \models [\forall]A \) iff \( M, w \models A \) for every world \( w \in W \). When \( \nabla A \) holds at \( x \) in \( M \), a witness for \( \nabla A \) at \( x \) in \( M \) is \( P \in \mathcal{P} \) such that \([x]_P \in P \). It may not be unique.

The logic of the class of all p-frames is axiomatizable and decidable. See Chapter 6 for axiomatic systems.

Logics of some other class of p-frames are also axiomatizable. For example, the logic of the class of all p-frames with its partition set being a singleton is axiomatizable with an additional characterization axiom: \((\nabla p \land \nabla q) \rightarrow [\forall](p \rightarrow q)\).

Nevertheless, not all logics on partitions are axiomatizable. Consider a subpartition relation on a set of partitions; i.e., a partition \( P \) is a subpartition of \( P' \) iff for any \( s \in P \) there is \( s' \in P' \) such that \( s \subseteq s' \). A p-frame is linear iff its partition set is linearly ordered with the subpartition relation. It is shown then that there is no axiom characterizing the class of all linear p-frames.
6. Complete answers versus complete and just complete answers

Philosophers repeat that human beings’ ability to reason is incomplete. An ordinal person would not see what happens even if there does exist total evidence indicating a symptom for those who have been trained. Nevertheless, it is not just a matter of reasoning ability. We need to be trained not just on how to reason from a given data but also how to obtain data and how to relate with other information. This is why I advocate to use partitionistic elimination.

6.1. ∇: some formal properties. See Chapter 6 for details.

- Non-monotonic: $A \rightarrow B$ does not infer $\nabla A \rightarrow \nabla B$.
- Non-antitonic: $A \rightarrow B$ does not infer $\nabla B \rightarrow \nabla A$
- $\nabla (A \lor B)$ is implied by $(\nabla A \land \nabla B)$, but not vice versa.
- $\nabla (A \rightarrow B)$ implies neither $\nabla \neg A$ nor $\nabla B$.

In the class of frames with the singleton partition set, $\nabla (A \rightarrow B) \land \neg \forall A \rightarrow \neg \nabla A$ is valid.

6.2. Adequacy with respect to intended applications. Let us examine some utterances to see if the notion of complete information can be applied to these cases.

Example 11. Implicative claims illustrate the case of giving complete and just complete information:

- If Alice went to the party, so did Bill.

Provided that he is sincere to give all the information he has, the speaker does want to commit himself to claim neither of the following:

- Alice did not go to the party.
- Bill went to the party.
Example 12. Another example of complete and just complete message is a promise:

- If it is not raining, we will go to the zoo.

Notice that $\nabla(A \rightarrow B)$ implies neither $\nabla A$ nor $\nabla B$, and $\nabla(A \rightarrow B)$ implies $\neg \nabla A$.

If an utterance of the promise bears a complete and just complete information, the utterer does not want to claim that it is raining, nor he claims that we are going to the zoo at the moment of the utterance.

Observations from the examples seem to lead to maximal informativeness conjecture:

Utterances to conduct a speech act bear just enough information.
Merely complete information is not adequate.

6.3. Previous usage of complete messages. While I would propose to take complete and just complete answers as only sufficient answers in each occasion, many authors focus on complete answers as paradigm of informative answers. Developing the theory of information content, for example, Dretske and Barwise exclude the problem from logic: incomplete reasoning ability. Dretske[16, p.14] analyzes:

Once something happens, we can take what did happen as a reduction of what could have happened to what did happen and obtain an appropriate measure or the amount of information associated with the result.

Furthermore, Dretske characterizes his notion of information content by non-triviality, a special case of counterfactuality in a non-temporal context:

\footnote{A similar phenomena is observed on imperatives in Chapter 3.}
A signal $r$ carries the information that $s$ is $F$ iff the conditional probability of $s$’s being $F$, given $r$ (and $k$), is 1 (but, given $k$ alone, less than 1) \[ [16, \text{p. 65}] \]

Notice that the message $r$ can be taken as a complete answer to a question which expects $s$’s being $F$. The message “John and Mary will come to the party this evening” will bring the information that Mary will come to the party. It is no problem with such a simple example. Nevertheless, the message “Sample gates are at Kirkwood and Indiana” brings the information where it is no matter what background knowledge its receiver has.

Imperfect reasoning ability can play the role of excuse with while we want make the formulation much closer to everyday phenomena\(^6\).

The idea of partitionism can be applied to an analysis of the possible world characterization of information content (Barwise):

To a person with prior knowledge $k$, $r$ being $F$ carries the information that $s$ is $G$ if in all the possible worlds compatible with $k$ and in which $r$ is $F$, $s$ is $G$ (and there is at least one possible world compatible with $k$ in which $s$ is not $G$).

The eliminative view, extracted in the inverse information principle, focuses on the note inside of the parentheses, which stipulates informativeness.

Moreover, the characterization of information content can be again restated in terms of partitionism, if the following assumptions are accepted:

(1) The body of prior knowledge $k$ agrees with the presuppositions of the context $C(k)$.

\(^6\)Barwise’s idea of constraints has the same point. Comparison and correspondence between constraints and multiple partitions will be a good project beyond this thesis.
(2) What carries information is a message. The message, whose content is that \( r \) is \( F \), decreases the presupposition set by eliminating possibilities incompatible with it. Let \( t(F) \) be the subset of \( C(k) \) where ‘\( r \) is \( F \)’ holds.

(3) The information that \( s \) is \( G \) in the definition of information content represents a piece of a focus in the partitionist characterization. The condition in the parentheses stipulates that the focus partition is not trivial. Let \( t(G) \) be the piece of the focus partition, or the proper subset of \( C(k) \) where ‘\( s \) is \( G \)’ holds.

Therefore, their approach to information contents takes complete messages as paradigmatic cases. It assumes that the receiver of the message can infer any consequences in the context. In general as seen in Example on page 26, however, getting local information about Ichigaya station is not enough to conclude that I am from Tokyo.

Such problematic situations require an amendment. There are at least two possible ways: either to control the set of possibilities or to manage the receiver’s inference ability.

The former is seen in Example 8 on page 29. In such an approach, epistemic possibilities may or may not reflect the actual world. These epistemic possibilities may not be simply taken as representations of the actual world, in particular, when changes of the world are described by any sort of possible worlds. The question how to associate those two sorts of possible worlds will require further investigation beyond the thesis.

An alternative following the latter amendment is the following partitionistic characterization of information content:

In the context \( C(k) \) represented by \( k \), the message \( r \) being \( F \) carries the information \( s \) being \( G \) is asked, if the focus partition is not trivial, and if \( t(F) \subseteq t(G) \).
In light of the notions of logic of questions and answers, it becomes clear that the definition characterizes the notion of a complete message, not a message with just enough information. It fails to exclude messages with too much information. No problem, as far as the interest is in how information flows. It is problematic only if our interest extends to the treatment of information excess.

6.4. **Only: some formal properties.** In fact, the logical property of complete information is reflected by the “only” operator. Gargov et al. [22] argue the logic of [ext] and [only], whose truth conditions are:

\[ M, w \models [\text{ext}] A \iff \text{for every } w' \text{ which is not } R_{ww'}, M, w' \models A \]

\[ M, w \models [\text{only}] A \iff \text{for every } w', \text{ if } M, w' \models A \text{ then } R_{ww'} \]

[only] has the following properties.

- Antitonic:

\[ A \rightarrow B \]

\[ [\text{only}] B \rightarrow [\text{only}] A \]

- [only](A ∨ B) is equivalent to the conjunction ([only]A ∧ [only]B) [De Morgan property]

Thus, [only](A → B) is equivalent to [only]¬A ∧ [only]B.

Nevertheless, as [only](A → B) is equivalent to [only]¬A ∧ [only]B, if [only] is used to represent insistence, upon the initial insistence he should insist both. Thus [only] should not be used for this representation purpose.

On the other hand, if ∇ represents a complete and just complete message, the initial claim entails only that he does not give complete and just complete information that Alice went to the party. This result is better than the [only] case.

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7 It is also suggested in van Benthem [65]. He proposes to read [ext]A as “¬A is strongly permitted” from the intuition (von Wright [68]) that strong permission of A or B should be equivalent to the conjunction of strong permission of A and of strong permission of B.
7. Classification of partial answers

There are three classes of partial answers. First, a loose reading of the definition of a partial answer allows such a situation that an answer is partial but not informative. The second class consists of answers which are partial and informative: a partial message in the first sense is such a message that is compatible in all the possible worlds in a direct answer or a piece of the partition, and no specification about the other possible worlds outside of the piece.

Third, as we use often in everyday situations, some partial answers are indeed disjunctions of complete and just complete answers. The category of partial messages is more specific. A partial message in the third sense is such a message that is a disjunction of direct answers, or compatible with all the possible worlds in some pieces of the partition, and incompatible with all the possible worlds in the remaining pieces.

Of course, partial messages perplex us. It does not seem to bring adequate information anyway, but which category do we tend to take to be natural? Consider this example: the question “which do you like, a soup, a salad, or an appetizer?” sets a context which can be represented by a partition with three pieces labeled with SOUP, SALAD, and APPETIZER. Non-complete answers can be:

- Soup or salad.
- Soup or caesar salad.
- Misoshiru or caesar salad.

It can be the case that the first falls in the third category, the second falls in the second, and the third does not fall in either.

Formal counterparts illustrate logical differences among the three notions. The first class, a standard S5 operator serves as a formal counterpart of logical properties
of answers which are partial but may or may not be informative. \([\text{dstit}]\) exactly corresponds to the second class of partial and informative answers\(^8\). The third class of unions of complete and just complete answers is also logically characterized by a new modal operator \([\text{pure}]\).

Independence of the logics is argued in Chapter 7, Section 7. Every axiom of the (single-agent) \([\text{dstit}]\) system is provable in the system of \([\text{pure}]\). On the other hand, the counterpart of the purity axiom in \([\text{dstit}]\) language is not valid in \([\text{dstit}]\) frames. Thus, the \([\text{dstit}]\) logic is a sublogic of the \([\text{pure}]\) logic. For more formal results, see Chapter 7.

Comparing their logical properties, the first and second subnotions, which rather seems to fit our intuition more than the third, still excludes the third category. Adequacy of the third sort of answers depends on whether such information excess can be accepted; if the questioner can serve misoshiru and caesar salad, such an answer will cause no problem, but it specifies the answerer's preference more than expected in the way of questioning.

8. Chapter summary

The idea of partitionistic elimination is applied to the logic of questions and answers. The proposal is to construct possible world semantics of operators complete, partial, and complete and just complete messages. The basic logic of those modal operators are axiomatizable. There are three subnotions of partial answers whose logical behavior differs.

Complete and just complete messages should be a paradigm of messages in efficient information inquiry. A paradigmatic use of merely complete messages leads to the logical omniscience problem. Behind the idea of complete message, there is an implicit

\(^8\)See pp. 103 – 105.
assumption of the receiver’s ability of reasoning. To avoid these sort of paradoxes of epistemic logic, it is necessary to explicitly describe acceptable messages instead of the idealistic assumption. The proposed method can model the receiver’s reasoning more accurately. A set of partitions of a possible world set will serve as a representation, and we can calculate relationships among sentences with respect to the receiver’s expectations and background knowledge.

9. Historical overview of logic of questions and answers

9.1. Emergence of logic of questions and answers. Philosophical logicians pointed out that interrogative sentences could be objects of formal logic, and strove toward the logical grammar of questions and answers. A forerunner already appeared in Cohen [14], where it is pointed out that questions should have logical form. Though his work shares topics with later research, Cohen had different motivations and aims. He characterized a question as a formula of the language of quantification logic with some free variables, and an answer to a question is a mere substitution instance. Prior-Prior [53] also pointed out the possibility of dealing with interrogative sentences in formal logic.

The first period of the research area came from the 1950s to the mid-1980s, focusing on fundamental conceptual analysis mainly by philosophical logicians. The first main figure is Hamblin[28]. The following are his theses on erotetic notions:

1. An answer to a question is a statement.
2. Knowing what counts as an answer is equivalent to knowing the question (Hamblin’s dictum).
3. The possible answers to a question are an exhaustive set of mutually exclusive possibilities.
David Harrah worked to capture the semantic notion of information in communication from the late-1950s to mid-1980s. He proposes several systems for more than twenty years\(^9\). Each of his systems is quite complicated and focuses different aspects from one to one. It seems hard to me to figure out how to unify those systems as the whole.

Nevertheless, Harrah’s ideas are historically important to influence Belnap, who worked on the issue from the late 1950s. They had regular discussions, which were summarized in forms of a preliminary report [5]. Its refined version appeared as Belnap-Steel[7]. Section 2 in this chapter summarizes his analysis. Belnap’s analysis in this step does not give any formal system; rather, it provides requirements for systems, which are to be constructed, to reflect the analysis. For example, he gives a taxonomy of answers, questions, and presuppositions, which any formal approach toward logic of questions and answers should be able to describe. The following period can be regarded as the period of implementation of his ideas and conceptual analysis.


9.2.1. Åqvist. Åqvist [1] follows the tracks of Hamblin’s logical analysis of imperatives, Belnap’s logic of questions and answers, and Hintikka’s epistemic logic. He is the first to explicitly formulate a question as a combined modality of imperative and epistemic modalities. According to his analysis, asking if A or not means the questioner does not know if A and she requests to erase the ignorance. It follows that an utterance of “is it A?” should be paraphrased as an utterance of “let me know if A”, a combination of an imperative modality ! (K modality) and an epistemic modality K (S4 modality).

\(^9\)[29], [30], [31], [32].
In Åqvist’s analysis, presuppositions of a question are defined as declarative sentences to give necessary conditions to follow the order requested by the question. Though it is slightly different from Belnap’s characterization of presuppositions of a question, the intended application is parallel: the definition gives classifications of questions and answers. For example, both Belnap and Åqvist call a question with an invalid presupposition risky. Åqvist pushes forward to propose the method of guarding, to avoid such situation by state all the presuppositions explicitly. Nevertheless, the idea does not fit his own formulation, since his formal language is finitary and he does not argue what to come when the set of presuppositions of a question is infinite.

Åqvist’s formalization takes four modalities: epistemic K (S4), imperative ! (K), and their duals. In particular, the permission modality i is the dual of imperative. Since all the modalities are normal, the formalization suffers from some problems. First, the problem of logical omniscience applies. Second, !Kp → ip is a theorem, which means under the intended interpretation that if you ask p, then p is morally permissible. It is counterintuitive and absurd.

9.2.2. Hintikka. Hintikka aims to apply his logic of questions and answers to scientific discovery, while others rather intend to construct formal semantics to influence the next period of implementation and linguistic applications. Hintikka \[34\] places the logic of questions and answers to a possible key to formal pragmatics, and argues toward the claim that his formulation of logic of questions and answer gives an example of formal pragmatics so that pragmatics can be formalized. His analysis seemingly adopts previous theories, especially the theory by Åqvist to claim that a question is a combination of imperative and epistemic modalities. Thus he paraphrases, for example, the question “which is \(···\)?” as “see to it that(I know \(p_1\) or \(···\) or I know \(p_k\))”, and called the disjunctive sentence inside of the parenthetical desiderata of the question.
Hintikka changes his mind when he begins to argue how to formalize those erotetic notions. He claims that, since imperative and optative depend on contexts, they are to be ignored in formulation. His focus is set only on the desiderata of a question. Then he moves on to concentrate on multiple-questions (questions with more than one interrogative word) to interpret an interrogative word as a combination of an epistemic operator and a quantifier. Thus, his formulation of the logic of questions and answers becomes an extended version of his game-theoretical semantics to involve epistemic operators.

Nevertheless, optative factors should not be ignored in analyzing questions. For example, presuppositions of a question are a mere condition for the existence of a true answer. Unless the questioner’s expectation is taken into consideration, the formulation would gain no plausible way to classify questions and answers with respect to presuppositions.

In short, Hintikka’s logic of questions and answers is a simplified version of previously existing theory by Belnap and Åqvist to pick up whatever fit to his own framework of epistemic logic and the game-theoretic interpretation of quantifiers. His treatment of multiple questions is inherited in the later approaches in the second period.

9.2.3. Polish and Chilean groups. There are several independent research groups in the area. Kubiński in Poland formulated logical relations among questions in terms of set theory. Wisniewski [70] followed the direction in 1990s.

There is another group of Stahl in Chile. His formulation is unique; given a logical system $L$, any $L$-consistent sentence is taken to be a direct answer. A complete answer is represented as a Boolean combination of direct answers\footnote{The notion is argued in Section 7.}, and a sufficient answer is
a sentence which implies a complete answer. A question is a set of sufficient answers. There seems no research in the group recently.

9.3. Moves in formal semantics. The second period, which intends to give a formal semantics for interrogative phrases of natural languages, began around the early 1980s. There are roughly four interrelated approaches: the $n$-ary relation approach toward applications of generalized quantifiers to the question issue, intensional logic approach as an extension of Montague semantics, situation semantics, and the dynamic approach. I will return to survey these approaches in the later section.

9.4. Partitionistic approach. Groenendijk and Stokhof ([26], [27]) formulate a question as a partition of possible worlds. It unifies previous treatments of interrogatives in formal semantics.

Groenendijk-Stokhof aims to reconstruct a categorial approach, which takes a question as a function associated with an underlying proposition. The idea of an underlying proposition comes from the traditional $n$-ary approach which is already found by Cohen [14], who takes a question as a predicate with free variables.

In a sense, their method improves the $n$-ary approach. It is said that there are at least two deficiencies in the traditional approach itself. First, since set-theoretical operations determines the interpretation of questions, it does allow permutation of the subject and the predicate. Another difficulty in the n-ary relation approach is from the approach with intensional logic: n-ary approach does not differentiate levels of questions. A single expression of the same structure could be used to express questions involving different levels of objects.

In Gronendijk-Stokhof’s treatment, underlying the question “Does John walk?” there is the proposition “John walks” to be evaluated at a possible world $w$, i.e., $\text{walk}(w)(j)$. The proposition itself is taken as a set of possible worlds $\{x : \text{walk}(x)(j)\}$. 
The question has two complete answers “John walks” (= \{ x : walk(x)(j) \}) and “John does not walk” (= \{ x : \neg walk(x)(j) \}). To answer the question is now paraphrased as: given a possible world \( z \), to identify to which set of possible worlds \( z \) belongs.

**9.5. Dynamic approach.** Van Rooy ([58], [56], [57]) argues against partitionistic approach to questions and answers. The basic idea is structural similarity between the Gronendijk-Stokhof model of a question as a set of answers and game-theoretical formulation of a decision problem as a set of strategies. His proposal is to identify a dominance relation on strategies with a relevance relation on interpretations of messages. A game-theoretical optimal is to be taken as the best interpretation. He furthermore extends the idea to non-partitionistic settings of questions to claim that a relevance relation generated from a non-partitionistic background structure results the mention-some interpretation as optimal, whereas a relevance relation generated from a partition yields the mention-all interpretation as optimal. His argument goes on to maintain that, since not all messages can be taken in the mention-all interpretation, partitionistic modeling of questions is not omnipotent; non-partitionistic modeling should be vindicated.

He alternatively proposes in [59] to relativize Gronendijk-Stokhof’s exhaustion operator with beliefs and preferences of agents, while Gronendijk-Stokhof allows only complete and just complete (“exhaustive” in van Rooy) in a single domain to induce non-monotonic logic (circumscription). \footnote{My proposal, argued later, aims to generalize Gronendijk-Stokhof by introducing multiple readings of a single interrogative for each agent.}

**9.6. Other approaches.**

9.6.1. Situation semantics. Ginzburg partly shows sympathy on the \( n \)-ary approach but points out that it lacks any connection to the world. Thus he develops his theory of questions and answers in the setting of situation semantics. His clue is
the notion of resolvedness. It extends the notion of direct answers in the first period
to be relativized with respect to an agent’s mental situation.

9.6.2. Intensional logic. Since the early 1980s, erotetic notions has been formulated in the setting of intensional logic. The direction has been refined and developed through the 1980s and 1990s.

The approach, based on Hamblin’s ideas, aims to express a question by a \( \lambda \)-abstraction. A semantic framework is the notion of partition of a space of possibilities. A question is semantically associated with a partition, which sets the presuppositions to be propositions which are true in all the available possibilities, and the set of answers as its members. It can be seen a formalized version of Belnap’s direction.

Vanderveken’s proposes a logic of speech acts in an extended form of intensional logic. It can be classified in the category since it should subsume logic of questions: asking a question is merely a kind of a speech act. Each speech act has a special modality polarized with six kinds of forces. Nevertheless, a big difference is its introduction of success and failure as kinds of truth values.
CHAPTER 3

Paradoxes of deontic logic revisited

This chapter aims to argue for the use of \([\text{unique}]\_\{\cdot\}\) as a deontic operator. Although the logic of \([\text{unique}]\_\{\cdot\}\) is neither a superlogic nor a sublogic of stit logics, it shares most properties with stit operators concerning traditional paradoxes of deontic logic. It is a nonmonotonic operator, and its logic evades the paradoxes of deontic logic as well as stit logics.

Section 6 will argue that philosophical differences from stit eventually arise when an additional moral principle is adopted.

1. Deontic logic

1.1. A historical survey. Deontic logic is a philosophical application of modal logic which takes permission and obligation as sentential operators. Obligation is considered moral necessity and permission moral possibility. Most analyses towards deontic logic admit the Kantian thesis that moral necessity should imply moral possibility, i.e., if doing A is a duty, then it is permitted. It leads to the definition: a system of standard deontic logic is a normal modal system with

\[ \square p \rightarrow \Diamond p \]

Thus any supersystem of the minimal system in which the condition, i.e., well-known formula D, is provable, is a system of standard deontic logic by definition. Often the system of standard deontic logic (SDL or D) stands for the minimal system that
satisfies the condition, i.e., the system D (the minimal normal system K plus the above formula as the additional axiom).

Nevertheless, some people take into consideration of Hume’s is-ought problem\(^1\): from what is (or is not), nothing about what ought to be (or ought not to be) can logically be concluded. For the supporters of the thesis, systems above T (in which \(\Box A \rightarrow A\) is derivable) are deductively so strong that they fail to satisfy Hume’s condition and do not reflect our intuition on the notion of ought. In any case, the system D of standard deontic logic serves as a popular system which satisfies those Kantian and Humean requirements.

**Notation 13.** Historically, \(\bigcirc\) is used for duty and obligation instead of \(\Box\), and \(P\) stands for permission instead of \(\Diamond\) in the context of deontic logic. \(\Box\) is supposed to stand for alethic necessity in such a context.

In the semantics, possible worlds are classified into the morally ideal worlds and the others, and obligation of a proposition \(A\) is formulated as \(A\) holds in every morally ideal world. The accessibility relation connects each world to every morally ideal world; the philosophical assumption that there must exist at least one morally ideal world turns out to correspond to the Kantian property of deontic notions that obligation implies permission, i.e., if \(A\) is obliged, then \(A\) must be possible.

**1.2. Paradoxes of deontic logic.** Nevertheless, it has been pointed out that the proposal of standard deontic logic leads to counterintuitive consequences. Prior ([50], [52]), Chisholm ([13]), and Lemmon ([41], [42], [49]) are in particular critical of the standard deontic logic to illustrate counterintuitive inferences in the formal system.

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\(^1\)Schutz [61].
• Paradoxes of derived obligation\textsuperscript{2}: the doing of what is forbidden commits us to the doing of anything whatsoever. Moreover, if the omission of any act is not permitted, i.e., if the act is obligatory, then we are ‘committed’ to it by any act whatsoever. It follows from the fact that $\neg PA \rightarrow \Box (A \rightarrow B)$ and $\Box B \rightarrow \Box (A \rightarrow B)$ are theorems of SDL.

• SDL is incapable of expressing the notion of contrary-to-duty imperatives\textsuperscript{3}: According to the divine law it is obligatory to will to repent of one’s sins, but it is forbidden to will to be guilty of a sin. Because being guilty of a sin is a necessary condition of repenting one’s sins, the latter cannot be efficaciously willed without willing the former except when one is already in a sin. \textsuperscript{4} Let $\Box A$ stand for one ought to see to it that $A$. Contrary to our expectation, all three do not fit our intuition, or they are “paradoxical”. $\Box (\neg A \rightarrow B)$ fails to express the contrary-to-duty imperative to see to it that $B$ in case we neglect our duty to see to it that $A$; for “no matter what state of affairs $B$ describes, it would be our duty to see to it that $B$, had we neglected our duty to see to it that $A$. This is absurd\textsuperscript{5}.” Another trial to let $\neg A \rightarrow \Box \neg B$ express the contrary-to-duty also fails; for its denial, which is supposed to be $\neg (\neg A \rightarrow \Box \neg B)$, entails $\neg A$; it means that denying that something is not a contrary-to-duty imperative logically entails that the subject. The third attempt by $\neg A \rightarrow \Box B$ also raises a problem (Chisholm [13]).

\textsuperscript{2}Prior [50]. It invoked von Wright’s reply [67] where he attempts to solve the problem in vain by relativizing the deontic notions.

\textsuperscript{3}In the modern deontic logic, it is presented by Chisholm. The analysis here follows von Wright [69].

\textsuperscript{4}The informal formulation is taken from Knuuttila [39, p.241], associated to Robert Rosetus in the fourteenth century deontic logic.

\textsuperscript{5}von Wright [69, p.107].
3. Paradoxes of Deontic Logic Revisited

- The good Samaritan paradox\(^6\): if anybody should help people who are in difficulty, he or she should be in difficulty.

- Ross’ paradox\(^7\): if you should post a letter, you should also post or burn it.

For \(\Box A \rightarrow \Box (A \lor B)\) is a theorem of SDL.

Thus, paradoxes of deontic logic also show discrepancies between our intuitive notions around agency and their formal counterparts. Some valid inferences of standard deontic logic should be invalid in our practical reasoning.

Here is a list of problematic formulas and rules, which can be read as paradoxes of deontic logic.

1. \(\neg PA \rightarrow \Box (A \rightarrow B)\): derived obligation, contrary to duty
2. \(\Box B \rightarrow \Box (A \rightarrow B)\)
3. \(\neg PA \rightarrow (A \rightarrow \Box B)\)
4. \(\neg (\neg A \rightarrow \Box \neg B) \rightarrow \neg A\): contrary to duty
5. \(\Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)\): Ross’ paradox (formula)

1.3. Logic of agency. The lesson from the obstacles for deontic logic proposed before 1980s, including standard deontic logic, is that the conceptual analysis of action\(^8\) should precede that of deontic notions. It actually fits our intuition, for a major everyday usage of “ought” concerns evaluation of human actions, not that of situations. In particular, if a deontic operator is motivated by game theory (for

\(^{6}\)Prior \([52]\) and Nowell-Smith and Lemmon \([49]\).

\(^{7}\)Ross \([60]\). As referred in his introduction to the article, he might have in mind Ernst Mally’s logic of imperatives \([47]\), where the paradox happens anyway, since the obligation modality is trivial \((\Box A \leftrightarrow A)\). Follesdal-Hilpinen \([18]\) illustrate Mally’s axiomatic system in detail in their historical survey of deontic logic and logic of imperatives.) The problem of triviality is solved by von Wright \([66]\)’s proposal of SDL, but the paradox still remains there.

\(^{8}\)Since deontic notions concern human actions and agency, logic of action in the current context, which should found any analysis of deontic notions, should reflect properties of agency and human actions. On the other hand, dynamic logic, often considered as logic of action, is intended to represent actions of machines, thus seems rather inappropriate for applications with deontic notions.
example, if it is designed to reflect logical properties of the best choice toward an action among available options), its truth conditions should involve evaluation of options by all possible means in a semantic approach. Syntactical characterization, on the other hand, of such operators should take an action sentence as its argument, since the deontic judgment is concerned with actions.

Thus:

- Action sentences in a given everyday language should be represented by sentences with a label so as to distinguish ought-to-do from ought-to-be in a syntactic level.
- Operators whose intended interpretations mention human actions, such as obligation, permission, and imperatives, should admit only sentences representing action sentences as their arguments.

1.4. Giving up monotonicity. It has been pointed out that the problematic formulas on page 50 fail to be theorems if the principle of monotonicity of $\Box$, or

\[
\vdash A \rightarrow B \quad \vdash \Box A \rightarrow \Box B
\]

is dropped off. Notice that non-monotonicity is just necessary, but not sufficient, for desirable deontic operators.

2. The see-to-it-that theory of agency

The see-to-it-that (stit) theory of logic of agency (Belnap et al. [6] ) intends to design a logic of agency which can serve as the basis of deontic logic. It satisfies the desiderata on page 51. With philosophical foundations in free will arguments, the theory proposes semantics of agency. The game-theoretically motivated notion of
choice is cast against branching semantics of indeterminist time (Thomason [64]); at each moment, each agent is assigned a set of available options.

In the stit theory, the basic idea to represent the notion of agency is the following prerequisites, which are supposed to guarantee philosophical applications by specifying the way of applying the formal results to philosophical discussions:

- An *agentive sentence* is a formula whose outermost operator is a stit operator (or a *stit sentence*). All and only agentive sentences formally represent action sentences in a given everyday language, and each action sentence in the everyday language is represented by an agentive sentence in the stit language\(^9\).

- When paraphrased in a modal approach, target concepts concerning human actions, such as duty and imperatives, are meaningful with an action sentence associated as a complement. Formalizing operators for the concepts can take only agentive sentences, or formulas whose outermost operator is a stit operator, as their arguments\(^10\).

- All stit formulas are meaningful as action sentences back in the intended application concerning human actions\(^11\).

For example, deontic logics based on stit focuses only on ought-to-do by letting stit sentences to be the only legitimate complements of deontic operators.

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\(^9\)Corresponding to “the agentiveness of stit thesis” and “the stit paraphrase thesis” in Belnap et al.

\(^10\)Corresponding to “the imperative content thesis”, “the restricted complement thesis”, and “stit normal form thesis”.

\(^11\)Originally the stit complement thesis.
There are several sorts of semantics and agency operators in the theory\textsuperscript{12}. Though the original structures both in Belnap and Horty, given on temporal structures, are complicated to remain faithful to philosophical intuition. An agent’s choice in the tree structure of indeterministic time is represented as a function to assign to each moment in the temporal structure a partition of possible worlds available at the moment; each piece represents a possible choice of the agent. Moreover, a real choice of doing $A$ should presuppose that $A$ is not settled at the moment; no possibility of non-$A$ means no real choice of $A$.

The argument here takes the simplest way on logic of agency to consider only possible world structures instead of Thomason’s branching time semantics in the original stit theory, since no temporal operator appears in the current argument.

2.1. Stit solution of paradoxes of deontic logic. Non-monotonicity is achieved already in logics in the stit theory of agency.

To make the argument below simple, let us focus on a single moment and a single agent $\alpha$. The truth condition of a so-called deliberative stit sentence $[\alpha \text{ stit: } A]$, which is to be read as a sentence representing $\alpha$’s doing $A$ — i.e., “$\alpha$ sees to it that $A$” —, is given in an S5 model $M = \langle W, R_\alpha \rangle$ as

$$M, w \models [\alpha \text{ stit: } A] \text{ iff for all } w'Rw' \Rightarrow M, w' \models A \text{ and there is } w'' \in W \text{ s.t. } M, w'' \not\models A$$

\textsuperscript{12}Several kinds of agency operators have been proposed in the stit theory: so-called achievement stit operators and deliberative stit operators. Achievement stit operators are supposed to reflect the intuition that an action is evaluated at the moment of its achievement, which is preceded by the moment of the agent’s choice toward the action. On the other hand, deliberative stit (dstit, in short) operators aim to capture agency where the action is evaluated at the moment of an agent’s choice. It is shown in [6] that both deliberative stit logic and achievement stit logic are axiomatizable, and that the latter is a sublogic of the former.
2.1.1. Deontic operators around stit. Various deontic operators based on stit theory have been proposed.

Horty [35], emphasizing game-theoretical motivations in the see-to-it-that (stit) theory of agency, proposes semantics of deontic operators with identifying duty as action according to dominant strategy. He introduces to stit semantics a utilitarian value assignment over branches, which represent possibilities at a moment, so as to define preference and dominance orderings among options at each moment. Defined with the dominance relations on choices, deontic operators are intended to label “actions at the dominant choice.” He proposes several deontic modalities including operators according to properties of value assignment; “⋯ holds in every dominant choice” and “⋯ holds in every optimal choice,” for example. Nevertheless, Murakami [48] argues that the resulting logic is normal, and does not essentially differ from the standard doentic logic.

Belnap and Bartha suggest that deliberative stit operators can be read as an imperative operator, identifying action simply as imperative or ought-to-do.

According to their proposal, \([ \alpha \text{ stit: } A ]\) is read as ‘Do A, α!’ . In fact, the imperative reading of dstit operators works nicely on Ross’s paradox. \([ \alpha \text{ stit: } \top ]\) is never true, since \(\top\) is true in every possibility so that the negative condition will never be fulfilled. A typical example of invalidity of the inference

\[
A \rightarrow B \\
\frac{}{[\alpha \text{ stit: } A] \rightarrow [\alpha \text{ stit: } B]}
\]

is given when the consequence \(B\) is settled true.

Since the deliberative stit modality is not monotonic, for example, Ross’ paradox

\[\text{They also propose Andersonian deontic operators with sanction in the stit framework.}\]
Post the letter!
Therefore, post or burn it!
is avoided.

Thus, the see-to-it-that (stit) logic of agency (Belnap et al. [6]) evades the previously existing paradoxes of deontic logic. For it avoids monotonicity by taking the counterfactual condition into consideration.

Nevertheless, the stit solution of Ross’ paradox has a further problem, when it is to incorporate with problems in ethical theories in the next section.

3. Obligation and ability

3.1. Moral dilemmas. Terrance McConnell\textsuperscript{14} characterize the notion of moral dilemmas:

What is common to the two well-known cases is conflict. In each case, an agent regards herself as having moral reasons to do each of two actions, but doing both actions is not possible. Ethicists have called situations like these moral dilemmas. The crucial features of a moral dilemma are these: the agent is required to do each of two (or more) actions; the agent can do each of the actions; but the agent cannot do both (or all) of the actions. The agent thus seems condemned to moral failure; no matter what she does, she will do something wrong (or fail to do something that she ought to do).

To avoid such conflict, it seems to useful to assume that ‘ought’ implies ‘can,’ or that, whenever the agent is required to do something, she is able to do it. Thus, it seems

intuitive to assume that imperative presupposes ability; no imperative of an agent’s doing $A$ can be intuitively meaningful with no possibility of her conducting $A$.

Nevertheless, the assumption with the following moral principles will infer a contradiction:

1. Principle of Deontic Consistency (PC): $\Diamond A \rightarrow \neg \Diamond \neg A$
2. Principle of Deontic Logic (PD): $\Box (A \rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B)$
3. Distribution of Deontic operator over Conjunction$^{15}$ (DC): $(\Diamond A \land \Diamond B) \rightarrow \Diamond (A \land B)$

McConnell summarizes that “[t]he most common response to the · · · argument was to deny PD.”

4. Replying to moral theory

Apparently McConnell’s conclusion coincides with the observation in deontic logic, since PD is the very formula of Ross’ paradox. Nevertheless, does any formal argument support the opinion with the additional condition that ‘ought’ implies ability?

Let us see how those principles are to be formulated in logic of agency, and what follows from each theory.

4.1. Principle of deontic consistency. The principle of deontic consistency in moral theory is also assumed in the framework of deontic logic in form of duality of obligation and permission. As most formal arguments will not work without it, discarding the principle is almost impossible.

4.2. Ability in logic of agency. Let us extend deontic logic to incorporate the argument by moral philosophers. For the purpose, we would like to introduce

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$^{15}$Let us tentatively call the principle DC. The converse follows from monotonicity.
the principle that ‘ought’ implies ability. Now, how the notion of ability is formally represented in the framework in logic of agency?

Logics of agency in the current argument assume that deontic operators should take an agentive sentence, or a sentence whose outermost operator is an agent operator.

An agent α’s ability of doing A at a moment m can be represented as there is a world w where α’s doing A holds at m. Using modal language $\mathcal{L}([\forall], \star, \Box)$ where $\star$ is an agent modality for α and $\Box$ for a deontic operator, it can be formalized as $\langle \exists \rangle \star A$.

Therefore, the principle that ‘ought’ implies ability is represented by the additional axiom: $\Box \star A \rightarrow \langle \exists \rangle \star A$.

4.3. Strengthened Ross’ paradoxes. With the assumption that ‘ought’ implies ‘be able to,’ the formal version of the inference (a stronger version of Ross’ paradox)

Post the letter! And post the letter and don’t burn it!

Therefore, post or burn it!

becomes valid in the stit treatment. Thus, it is unavoidable with the deliberative stit.

It means that merely giving up monotonicity as in stit is insufficient to avoid the sort of paradox, as far as it is assumed that ‘ought’ implies ‘be able to’.

5. Proposal of partitionistic elimination

A proposal here is again to apply partitionistic elimination. Taking the all-and-only modality as an imperative modality, the undesired inference becomes invalid.

$[\text{unique}]_{\downarrow} A$ can be paraphrased in a similar way to the stit case as: $[\text{unique}]_{\downarrow} A$ iff $\Box A \land \lnot \lnot A$, read as “A holds under the choice and A is strongly permitted.”

In fact, with imperatives represented by $[\text{unique}]_{\downarrow}$, the second counterintuitive inference of the strengthened Ross’ paradox now becomes invalid.
Moreover, it invalidates the following inference, too:

- Post letter A and letter B! Therefore, post letter A!

Thus, taking the notion of strong permission seriously with the all-and-only modality, the proposal here improves deontic logic.

5.1. Alternative ideas. We have mentioned the modality [only] on page 37. What if we use [only] instead to represent imperatives? In fact, as [only]A does not imply [only](A ∨ B), the Ross’ paradox is evaded. Nevertheless, it validates this absurd inference:

- Post or burn it! Therefore, post it and burn it!

Thus, [only] can’t be interpreted as an imperative indicator.

Another trial is to let ¬[only]¬ represent imperatives, since [only] behaves much like permission. Ross’ paradox is invalidated along with the line, in fact. It will not work, however, as orders of impossibles are always validated. Thus, [unique]{·} fits our purpose more than [only]. Information excess should not be overlooked.

6. Discussion

When the principle that ought implies ability is adopted, my theory goes a different way from the stit theory. The stit theory gives up distribution of conjunction over deontic operator, while mine invalidates Ross’ paradox no matter whether or not the additional principle is imported. Table on page 58 summarizes differences between the deliberative stit operator and [unique]{·}.

<table>
<thead>
<tr>
<th>Distribution of ∧ over obligation</th>
<th>stit</th>
<th>[unique]{·}</th>
</tr>
</thead>
<tbody>
<tr>
<td>invalid</td>
<td>yes</td>
<td>invalid</td>
</tr>
<tr>
<td>PD with moral dilemma</td>
<td>valid</td>
<td>invalid</td>
</tr>
</tbody>
</table>

Table 1. Summary of stit and [unique]{·} concerning moral dilemma
It seems to be a matter of choice which operator to use. Nevertheless, if the previous arguments from the paradoxes of deontic logic are really to be considered, the paradoxes are to be evaded in any extension which are consistent with arguments in moral theory. Is not a theory shaky if it behaves differently with an additional principle? Thus, I would like to say that my theory is a better continuation of the project of deontic logic in general.
CHAPTER 4

Summary

This thesis has argued the applications of the proposal of partitionistic elimination. It suggests to us using a partition to decide how many possibilities to eliminate. Elimination of possible worlds should be relativized to available partitions. Elimination of possibilities is significant in a context (represented as a set of partitions of possibilities) when it leaves only one piece in a non-trivial partition in the context.

On the basis of the proposal, some logics of partitionistic modal operators are investigated. The basic logic is axiomatizable, and some axioms correspond to frame conditions. Not all superlogics are axiomatizable, however; the frame condition of nested partitions is non-axiomatizable.

In application to logic of questions and answers, partitionistic modal operators are interpreted as complete and just complete answers. The main idea of this thesis, partitionistic elimination, suggests that complete and just complete answers should be taken as the only legitimate answers in each occasion. Focusing on complete answers leads to variants of logical omniscience; to avoid it, we need specification of a legitimate set of complete and just complete answers. It can be semantically formalized with multiple partitions, and the resulting modal logics are non-monotonic so that they avoid logical omniscience.

The notion of partial answers also can be formalized. I propose to refine the notion of partial answer into three categories. First, partial but possibly not informative answers; second, partial and informative answers; and third, partial answers which are disjunctions of complete and just complete answers.
The other application of partitionistic operators are to deontic logic. The thesis argues a semantic condition motivated from moral dilemma arguments. Adding the condition to the semantics and partitionistic modal logic and that of stit logic suggests philosophical differences between those approaches toward agency and deontic logic.

The fact that the single proposal can be applied to information and obligation suggests parallelism between declaratives and imperatives. Although Hintikka [9, pp.340] states that possible worlds simply connect our knowledge and action, however, it might not be that simple. I suspect that epistemic and causal usages of possible worlds would not be able to provide a good framework if they are just jumbled together. Mixing up both sorts of possible worlds is no less than muddling up toy coins with real coins. Each sort obviously is in a different status to be used in a different system. A preliminary work on each kind of possible worlds should be done before investigation of interactions of those two sorts. This line of analysis should be clarified before formulation of any notion of strategy, because any strategies have to take into consideration both possible actions and information reflecting the consequences of actions. The thesis targets the preliminary task and leaves the latter for future work, however.

I would like to close the philosophical part with the following citation from Bar-Hillel [3, pp.148-149]:

There exists no logic that covers all English imperative sentences, just as there exists no logic that covers all English declarative sentences. But just as there exists a logic of statements made by uttering English declarative sentences underdeveloped as it may be, so there exists a logic of commands (and of instructions, and of encouragements, etc.) issued by uttering English imperative sentences,
though this logic has hardly been developed at all, partly due to misconceptions of the type illustrated above. In order to develop such a logic formally — and there seems to exist no other way of doing it that deserves serious attention — the commands (and statements) have to be presented first in some normalized form, preferably in some formalized language, but at least in some ‘natural’ language that has been sterilized and exempted from all disturbing pragmatic account — expanding it would require a volume and would be worth one). This, of course, now poses the double problem of, firstly, specifying the exact nature of this normalization and, secondly, and still more formidably, establishing the ‘rules of normalization’, governing the transformation from ordinary speech into the normalized language. The second task can surely not be performed by logicians as such. One should, therefore, be surprised to find clever logicians writing foolish papers when they succumb to the temptation of not paying slightly sufficient attention to certain distinctions in their treatment of natural languages just because these distinctions are ematerial for language systems.

I believe that this thesis is not merely foolish.
CHAPTER 5

A toolkit of modal logic

This chapter surveys technical backgrounds of the arguments in Chapters 6 and 7. Section 1 describes definitions in syntax and semantics. Section 2 discusses basic completeness proof of modal logic. Section 3 illustrates the idea of bisimulation. Section 4 describes a standard method to show decidability of modal logic. Another method via decidability of a fragment of the second order logic is also argued in Section 5. A technique to deal with the difference operator is presented in Section 6. Its completeness proof can be compared with that of modal logic on partitions in Chapter 6. Section 7 surveys neighborhood semantics in order to compare with semantics proposed in Chapter 6.

1. Syntax and semantics

A *modal language* A modal signature \( L \) consists of

(1) A denumerable set Prop of propositional variables: \( p_1, p_2, \cdots \);

(2) Boolean connectives: \( \land, \neg \); and

(3) unary operator symbols.

When \( Op \) is the set of unary operator symbols of a modal language, let \( L(Op) \) denote the language. Let \( \mathcal{L}, \mathcal{L}', \cdots \) stand for metavariables for modal languages, and \( *, \cdots, *_n \) metavariables for unary operators. In particular, when \( *, \cdots, *_n \) are unary operator symbols of a modal language, \( L(*_1, \cdots, *_n) \) denotes the resulting language.

The set of *sentences of* \( L(Op) \) is recursively defined as usual:

(1) A propositional variable is a sentence of \( L(Op) \).
(2) When \( A \) and \( B \) are sentences of \( L(Op) \), so are \( \neg A \) and \( A \land B \).

(3) When \( A \) is a sentence of \( L(Op) \) and \( \star \) is a unary operator symbols of \( L(Op) \), \( \star A \) is a sentence of \( L(Op) \).

\( \star^n \) is an abbreviation of a run of \( n \) \( \star \)'s: i.e., \( \star^0 A = A \) and \( \star^{i+1} A = \star \star^i A \). Define

\[
\begin{align*}
\star^{-} X & = \{ A : \star A \in X \} \\
\star^{+} X & = \{ \star A : A \in X \}
\end{align*}
\]

When \( X \) is a finite set of formulas \( A_1, \ldots, A_n \), \( \bigwedge X \) is an abbreviation of \( A_1 \land \cdots \land A_n \), and \( \bigvee X \) is of \( A_1 \lor \cdots \lor A_n \). The symbols \( \Box \) and its dual \( \Diamond = \neg \Box \neg \) will stand for standard modal operators.

First, semantic definitions for a monomodal language are given as follows. A \( L(\Box) \)-frame \( F \) is a pair \( \langle W, R \rangle \), where \( W \) is a set and \( R \) is a binary relation on \( W \). A truth-value assignment on \( L(\Box) \)-frame \( F \) is a function from \( \text{Prop} \) to powerset of \( W \), \( \wp W \). A \( L(\Box) \)-model on a \( L(\Box) \)-frame \( F = \langle W, R, v \rangle \) is a triple \( \langle W, R, v \rangle \), where \( v \) is a truth-value assignment on \( F \). The truth conditions for \( L(\Box) \)-sentences are:

1. \( M, w \models p \) iff \( w \in v(p) \) for a propositional variable \( p \)
2. \( M, w \models \neg A \) iff not \( M, w \models A \)
3. \( M, w \models A \land B \) iff \( M, w \models A \) and \( M, w \models B \)
4. \( M, w \models \Box A \) iff for each \( w' \) such that \( wRw' \), \( M, w' \models A \)

With truth at a world in a \( L(Op) \)-model defined, the following notions of validity are defined: \( A \) is valid in a \( L(Op) \)-model \( M \) (written \( M \models_{L(Op)} A \)) if, for any \( w \) in \( W \), \( M, w \models_{L(Op)} A \); \( A \) is valid in a \( L(Op) \)-frame \( F \) (written \( F \models_{L(Op)} A \)) if, for any model \( M \) on \( F \), \( M \models_{L(Op)} A \); and \( A \) is \( L(Op) \)-valid (written \( \models_{L(Op)} A \)) if, for any frame \( F \), \( F \models_{L(Op)} A \).

An axiomatic system \( S \) on language \( L \) consists of a set of \( L \)-formulas (axioms) and a set of pairs of a set of \( L \)-formulas and an \( L \) formula (rules of inference).
A theorem of an axiomatic system $S$ is defined recursively: i.e., an axiom of $S$ is a theorem; and, if $X$ is a set of theorems and $(X,A)$ is a rule of inference, $A$ is a theorem. Let $\vdash_S A$ stand for “$A$ is a theorem of an axiomatic system $S.$” When the system in question is clear, the subscript is omitted.

A logic $\mathcal{L}$ on language $\mathcal{L}$ is a set of $\mathcal{L}$-formulas. An axiomatic system $S$ axiomatizes a logic $\mathcal{L}$ if the set of theorems of $S$ is identical to $\mathcal{L}$. In such cases, often $\mathcal{L}$ is identified with $S$. A logic $\mathcal{L}$ is axiomatizable if there is an axiomatic system $S$ which axiomatizes $\mathcal{L}$.

Let $X$ be a set of formulas. $X \vdash_S A$ iff there is a finite subset $X' \subseteq_{\text{fin}} X$ such that $\vdash_S \bigwedge X' \rightarrow A$.

Let $\bot$ be an abbreviation of $p \land \neg p$ for any $p \in \text{Prop}$, and $\top$ be of $p \lor \neg p$. A set $X$ of formulas is consistent in an axiomatic system $S$ if $X \not\vdash_S \bot$. Otherwise, $X$ is inconsistent in $S$.

A consistent set $X$ of formulas is called maximal iff for each consistent set $Y$ such that $X \subseteq Y, Y \subseteq X$. A set of formulas is complete if, for each formula $A$, either $A$ or $\neg A$ belongs to the set.

2. Completeness and basic correspondence results

A modal logic $\mathcal{L}$ is complete for a class $\mathcal{C}$ of frames if $\mathcal{L} = \bigcap_{F \in \mathcal{C}} \{A : F \models A\}$.

The following logical system $K$ is complete for the class of all modal frames.

- axioms and rules of inference (modus ponens and substitution) of classical propositional logic.
- $\Box_i(p \rightarrow q) \rightarrow (\Box_i p \rightarrow \Box_i q)$.
- If $A$ is a theorem, then so is $\Box_i A$.

A Henkin-style proof, outlined below, is a standard completeness proof for many modal systems.
Lemma 14. A $S$-consistent set $X$ of formulas can be extended to a maximal $S$-consistent set.

Proof. Enumerate all the formulas $A_0, A_1, \cdots$, and let $X_0 = X$. Let $X_{n+1} = X_n \cup \{A_n\}$ if $X_n \cup \{A_n\}$ is $S$-consistent; otherwise, let $X_{n+1} = X_n$. Define $Y = \bigcup_{i<\omega} X_i$. It is easy to check that $Y$ is maximal $S$-consistent. □

Observation 15. Let $X$ be maximal consistent.

1. For each formula $A$, either $A \in X$ or $\neg A \in X$, and not both.
2. Every axiom belongs to $X$.
3. If $\vdash A \rightarrow B$ and $A \in X$, $B \in X$.
4. Every theorem belongs to $X$.

Lemma 16. For any consistent set $X$, $\square^{-}X$ is consistent.

Proof. If not, there is a finite subset $Y \subseteq_{\text{fin}} \square^{-}X$ such that $\vdash \bigwedge Y \rightarrow \bot$. It follows that $\vdash \square \bigwedge Y \rightarrow \bot$, which is equivalent to $\vdash \bigwedge \square Y \rightarrow \bot$. Yet this contradicts the assumption that $X$ is consistent. □

Lemma 17. Let $X, Y$ be maximal consistent sets. $\square^{-}X \subseteq Y$ iff $\Diamond^{+}Y \subseteq X$.

Proof. Suppose $\square^{-}X \subseteq Y$. Let $A \in Y$. If $\Diamond A \not\in X$, $\square \neg A \in X$. Therefore, $\neg A \in Y$, which is contradictory. Thus $\Diamond A \in X$.

Suppose $\Diamond^{+}Y \subseteq X$ and let $\square A \in X$. If $A \not\in Y$ then $\Diamond \neg A \in X$. It is equivalent to $\neg \square A \in X$, which is a contradiction. Thus, $A \in Y$. □

Given a consistent formula $A$, a canonical model of $A$ can be constructed as follows. Let $W^c$ be the class of maximal consistent sets. Define $R^c$ as: $R^c XY$ iff $\square X \subseteq Y$. Define $v^c$ as $v^c(p) = \{X : p \in X\}$. Let $M^c = \langle W^c, R^c, v^c \rangle$.

Lemma 18. For any formula $B$ and $X \in W^c$, $M^c, X \models B$ iff $B \in X$. 

COROLLARY 19. $K$ is complete.

A class $C$ of frames is defined by a first-order formula $\phi$ if, with $W$ taken as a domain and $R$ as a binary relation, $\phi$ is valid in every frame in $C$.

A modal formula $A$ corresponds to a frame condition $\phi$ if it is valid all and only in the frames in the class defined by $\phi$. The table summarizes some of correspondence results; note that not every modal formula corresponds to a frame condition, nor every first-order condition has a modal formula corresponding to it.

<table>
<thead>
<tr>
<th>Name</th>
<th>Modal formula</th>
<th>Defining formula</th>
<th>English name$^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>$\square p \rightarrow p$</td>
<td>$\forall x (Rxx)$</td>
<td>reflexivity</td>
</tr>
<tr>
<td>B</td>
<td>$p \rightarrow \square \diamond p$</td>
<td>$\forall xy (Rxy \rightarrow Ryx)$</td>
<td>symmetry</td>
</tr>
<tr>
<td>4</td>
<td>$\square p \rightarrow \square \square p$</td>
<td>$\forall xyz (Rxy \land Ryz \rightarrow Rxz)$</td>
<td>transitivity</td>
</tr>
<tr>
<td>5</td>
<td>$\diamond p \rightarrow \square \diamond p$</td>
<td>$\forall xyz (Rxy \land Rzx \rightarrow Ryz)$</td>
<td>euclidean</td>
</tr>
</tbody>
</table>

3. Bisimulation

Bisimulation provides a tool in comparing structures, unifying traditional tools for semantical investigations in modal logic, such as generated submodels, bulldozing, and $p$-morphism$^2$. Once two structures are proved to be bisimilar, it is known that they support the same formulas.

As bisimulation is useful in organizing later arguments in this chapter, let us summarize the definition and the main lemma (bisimulation lemma).

DEFINITION 20. Two modal frames $F = \langle W, R \rangle$ and $F' = \langle W', R' \rangle$ are bisimilar via a bisimilarity relation $Z \subseteq W \times W'$ iff

(1) If $Rxy$ and $Zxx'$, there is $y' \in W'$ such that $R'x'y'$ and $Zyy'$.

$^2$See [19] for details.
(2) If $R'x'y'$ and $Zxx'$, there is $y \in W$ such that $Rxy$ and $Zyy'$.

Two modal models $M = \langle F, v \rangle$ and $M' = \langle F', v' \rangle$ are bisimilar via $Z$ iff

1. $F$ and $F'$ are bisimilar via $Z$
2. For any propositional variable $p$, $w \in W$ and $w' \in W'$, of $Zww' w \in v(p)$ iff $w' \in v^*(p)$.

**Lemma 21.** (Bisimulation lemma) Let two modal models $M = \langle F, v \rangle$ and $M' = \langle F', v' \rangle$ be bisimilar via $Z$ and suppose $Zxx'$. Then for any formula $\phi$, $M, x \models \phi$ iff $M', x' \models \phi$.

**Proof.** By induction. Here is the induction step for a formula whose outermost operator is $\Box$: Suppose $M', x' \not\models \Box \phi$. By truth definition, there is $y'$ such that $M', y' \not\models \phi$. Then there is $y$ such that $Zyy'$, $Rxy$, and $M, y \not\models \phi$. Thus, $M, x \not\models \Box \phi$. The converse is similar. \qed

**Definition 22.** $\langle W', R', o \rangle$ is called a generated subframe of a frame $F = \langle W, R \rangle$ generated by $o \in W$, where

1. $W' = \{w \in W : \text{there is } n \in N \text{ such that } R^n ow\}$.
2. $R'$ is the restriction of $R$ on $W'$.

$M' = \langle W', R', o, v' \rangle$ is a generated submodel of a model $M = \langle W, R, v \rangle$ by $o$ if $\langle W', R', o \rangle$ is a generated frame of $\langle W, R \rangle$ by $o$ and $v'$ is a valuation such that $v'(p) = v(p) \cap W'$.

**Proposition 23.** Let $M' = \langle W', R', o, v' \rangle$ be a generated submodel of $M = \langle W, R, v \rangle$ by $o$. Then for any formula $\phi$ and $w \in W'$, $M, w \models \phi$ iff $M', w \models \phi$.

**Proof.** The equality relation on worlds serves as a bisimulation relation. \qed

**Corollary 24.** For any formula $\phi$, $\phi$ is valid in $M$ iff $\phi$ is valid in $M'$. 
4. Decidability: filtration and reduction

In investigations of a given logic, the first question is whether it is complete for any axiomatic system. The second question is decidability: is there any procedure to decide whether a formula belongs to the logic? For modal logic, there are two main tools for decidability: filtration and reduction.

Filtration is a method to show decidability via constructing a finite model of a given consistent formula. If a system has the finite model property (f.m.p.) i.e., a consistent formula has a finite model, you can effectively find a countermodel for any non-valid formula. Combined with axiomatizability, f.m.p. implies decidability. Nevertheless, filtration is limited to those with the finite model property; moreover, it sometimes requires creativity to set an ad hoc filtration structure.

Reduction is a more powerful method than filtration to systematically prove that a system is decidable by reducing the problem to decidability of a fragment of second-order logic, invoking Rabin’s theorem on decidability of the second-order theory on a tree.

4.1. Decidability via filtration: introduction. The filtration method\(^3\) works for many logics with the finite model property. The idea behind the method is this: if you can recursively enumerate the theorems of a logical system which axiomatizes the logic in question, and if you can recursively enumerate finite models of the logic, you will eventually decide if any formula is a theorem or has a countermodel. Thus, a logic not recursively axiomatizable may be undecidable even if it has the finite model property.

\(^3\)Hughes and Cresswell[36](ch. 8) surveys the filtration method. Blackburn et al.[8] (ch. 6, sec. 2) summarizes the finite model method in connection to axiomatizability.
4.2. How filtration works. Completeness is often proved by construction of a canonical model of a consistent formula. Given a canonical model $M = \langle W, R, v \rangle$, construct a finite model via a filtration set by making a quotient model as follows. A filtration set $\Gamma$ should be taken as a finite set closed under subformulas of the formula in question (i.e., the given consistent formula in the original canonical model construction). Introduce an equivalence relation $\cong_\Gamma$ on the class of maximal consistent sets as:

$$w \cong_\Gamma w' \iff, \text{for any } A \in \Gamma, \ A \in w \iff A \in w'.$$

Define $[M]_\Gamma = \langle [W]_\Gamma, [R]_\Gamma, [v]_\Gamma \rangle$, where (1) $[W]_\Gamma = \{ [w]_\Gamma : w \in W \}$, (2) $[R]_\Gamma$ adequately (see below), and (3) $[v]_\Gamma(p) = \{ [w]_\Gamma : w \in v(p) \}$ for any propositional variable $p \in \Gamma$; otherwise, let $[v]_\Gamma(p) = \emptyset$. As all worlds in each $[w]$ agree on $\Gamma$, $[v]_\Gamma$ is well-defined. The finite model property will be shown once it is shown that $M$ and $[M]_\Gamma$ support the same formulas in $\Gamma$.

A filtration relation is not uniquely determined. In general, any relation satisfying the first condition of the above may serve as a filtration relation via $\Gamma$: for any $w, w' \in W$,

1. if there is $y \in W$ such that $Rwy$ and $y \cong_\Gamma w'$, then $[R]_\Gamma[w]_\Gamma[w']_\Gamma$.

2. if $[R]_\Gamma[w]_\Gamma[w']_\Gamma$ then, for every $\Box A \in \Gamma$, if $M, w \models \Box A$ then $M, w' \models A$.

Among such relations, there are finest and coarsest. The finest filtration relation $[R]_\Gamma^f$ is a relation satisfying: for any $w, w' \in W$,

1. if there is $y \in W$ such that $Rwy$ and $y \cong_\Gamma w'$, then $[R]_\Gamma^f[w]_\Gamma[w']_\Gamma$.

2. the converse of the above: if $[R]_\Gamma^f[w]_\Gamma[w']_\Gamma$ then there is $y \in W$ such that $Rwy$ and $y \cong_\Gamma w'$.

The coarsest filtration relation $[R]_\Gamma^c$ via $\Gamma$ is a relation satisfying: for any $w, w' \in W$,

1. if $[R]_\Gamma^c[w]_\Gamma[w']_\Gamma$ then, for every $\Box A \in \Gamma$, if $M, w \models \Box A$ then $M, w' \models A$. 
(2) its converse: Suppose for every $\square A \in \Gamma$, if $M, w \models \square A$ then $M, w' \models A$;

Then, $[R]_\Gamma^f[w]_\Gamma[w']_\Gamma$

It is easy to show that, for any relation satisfying the above condition, $[R]^f_\Gamma \subseteq [R]_\Gamma \subseteq [R]^c_\Gamma$.

**Proposition 25.** Let $M = \langle W, R, v \rangle$ and its finest filtration model $[M]^f_\Gamma = \langle [W]_\Gamma, [R]^f_\Gamma, [v]_\Gamma \rangle$ via $\Gamma$. Then, $M$ and $[M]_\Gamma$ support the same formulas among $\Gamma$.

**Proof.** By induction. Here we check the $\square$ case. Suppose $M, w \not\models \square A$. Then there is $y$ such that $Rwy$ and $M, y \not\models A$. By induction hypothesis, it follows that there is $[y]$ such that $[R][w][y]$ and $[M], [y] \not\models A$. Thus, $[M], [w] \not\models \square A$. Conversely, Assume $[M], [w] \not\models \square A$. Then, there is $[y]$ such that $[R][w][y]$ and $[M], [y] \not\models A$. By induction hypothesis, $M, y \not\models A$. By (the contraposition of) filtration condition, $M, w \not\models \square A$.

For the case of finest filtration, bisimulation also works.

**Proposition 26.** Let $M = \langle W, R, v \rangle$ and its finest filtration model $[M]^f_\Gamma = \langle [W]_\Gamma, [R]^f_\Gamma, [v]_\Gamma \rangle$ via $\Gamma$. Then, $M$ and $[M]_\Gamma$ support the same formulas among $\Gamma$.

**Proof.** $F = \langle W, R \rangle$ and its filtration frame $[F]_\Gamma = \langle [W]_\Gamma, [R]_\Gamma \rangle$ is bisimilar via $w \mapsto [w]$. A simple induction shows that $M$ and $[M]_\Gamma$ support the same formulas among $\Gamma$.

**4.3. Examples.** The filtration method applies to most of famous modal systems; in fact, $K$, $\mathbf{T}$, $\mathbf{B}$, $\mathbf{4}$, $\mathbf{S4}$ (the system with additional axioms $\mathbf{T}$ and $\mathbf{4}$), and $\mathbf{S5}$ (the system with $\mathbf{T}$ and $\mathbf{5}$) among other systems are shown via the method to be has the finite model property. Being finitely axiomatizable, each of the systems is decidable. Here are some examples.
4.3.1. $K$. $K$ has the finite model property. Given a consistent formula, take the set of its subformulas as a filtration set, and apply filtration to its canonical model. As every filtration model is a $K$-model, $K$ has the finite model property.

4.3.2. $S4$. $S4$ has the finite model property, too. Use the same filtration set as the $K$ case, but take the coarsest filtration relation as a filtration relation $[R]$. Then, as $[R]$ is transitive and reflexive, the filtration model is an $S4$ model and it is finite. Thus, $S4$ has the finite model property.

5. Decidability: a general reduction method

Decidability of many modal systems can be reduced to decidability of a fragment of second-order logic, using Rabin’s theorem on decidability of the second-order theory on tree. Rabin’s theorem is strong and difficult to proof; a proof is beyond the purpose of this chapter; a proof appears in Zeitman[73]. Gabbay[21] presents a unified method of reduction for regular modal logics. Blackburn et al.[8] illustrates applications.

This is a general scheme of the reduction method for decidability$^4$.

**Definition 27.**

A class $C$ of structures reduces a class $C'$ if

1. there is $C'_0 \subseteq C'$ and a map $\nu : C'_0 \to C$ such that for all $S \in C$ there is $S' \in C'_0$ such that $\nu(S')$ is isomorphic to $S$.

2. there is a sentence $\psi$ which is valid in every structure in $C$ and only in the structure in $C$.

3. there is an effective translation $\tau$ of sentences from the language of $C$ to that of $C'$ such that $S \models \varphi$ iff for all $S'^{\dagger}$ such that $S'^{\dagger} \cong \nu(S)$, $S'^{\dagger} \models \tau(\varphi)$.

$^4$Gabbay[20], 15.3.3.
Theorem 28. Suppose that $C$ reduces to $C'$, and that the theory of $C'$ is decidable. Then the theory of $C$ is decidable.

Proof. Let $A$ be an algorithm for $C'$. As $\psi$ defines $C$, $A$ is valid in $C$ iff $\tau(A) \land \psi$ is valid in $C'$. As $\tau$ is effective and checking validity of $\tau(A) \land \psi$ can be done by $A$. Thus, validity of $A$ is decidable. \hfill $\square$

Corollary 29. Suppose there is a sentence $\psi$ such that $C_0 = \{F \in C : F \models \psi\}$. Then, $C_0$ is decidable if $C$ is.

5.1. A reduction method for monomodal logic. Gabbay\[21\] describes the reduction method in the form applicable to regular modal logics. Here given is a simplified form for normal modal logics. It basically follows Gabbay, but takes every world as normal.

A basic idea behind the method is that a modal structure can be taken as a first-order structure, with a propositional variable $p_n$ taken as a unary predicate $P_n$ as its counterpart. An accessibility relation is taken as a binary relation on the first-order domain. In other words, a relational structure can be associated with each of two signatures, one of modal logic and the other of first-order logic.

Since evaluation of a modal formula at a world depends only on evaluation of its subformulas at worlds accessible within a finite number of steps from the original world, it is a powerful method for modal logics to reduce a decidability problem to Rabin’s decidability result of second-order theory of the tree of finite sequences of natural numbers.

5.2. Rabin’s theorem. Let $X$ be a set of natural numbers, either $\omega$ or $X = n = \{i : i < n\}$. Consider the set $X^*$ of all finite sequences on $X$, including the empty

---

\[ Related \ material: \ [20]\ summarizes \ decidability \ results \ of \ temporal \ logics \ by \ reduction \ to \ theorems \ on \ second-order \ logic, \ such \ as \ of \ Shelah \ and \ Büchi. \]
sequence $\epsilon$. Let $\circ$ stand for concatenation. For each natural number $i \in X$, the $i$-th successor function is $s_i(t) = t \circ i$.

For each $t, s \in X^*$, define $t \leq s$ iff there exists $u \in X$ such that $s = t \circ u$. $t < s$ iff $t \leq s$ and $t \neq s$. A lexicographical order $<_l$ on $X^*$ is defined as $t <_l s$ iff either $t <_l s$, or $t = t_0 \circ n \circ t_1 <_l t_0 \circ m \circ t_2 = s$ and $n < m$.

The tree $T_X$ on $X$ is the structure $\langle X^*, (s_i)_{i \in X}, \leq, <_l \rangle$. The monadic second-order theory of $n$-successor functions ($SXS$) is the theory of the tree on $X$.

**Theorem 30.** (Rabin’s theorem) $S\omega S$ is decidable.

### 5.3. Reduction: a mapping to tree structure

We are to consider the class $C$ of frames of a modal logic with the propositional monomodal language and the class $C^*$ of subtrees of $T_\omega$ whose language is that of second-order logic.

Let $C$ be a class of generated frames $F = \langle W, R, o \rangle$. A first-order formula $G$ defines $C$ if $C$ is the class of all first-order structure $\langle W, R, o \rangle$ (with $R$ and $o$ taken as first-order predicates) in which $G$ is valid.

Given a generated frame $F = \langle W, R, o \rangle$, consider another generated frame $F^* = \langle W^*, R^*, o^* \rangle$ arises, where $W^*$ is the set of finite sequences of form $(o, x_1, \cdots, x_m)$ whereas $R x_i x_{i+1}$ holds for each $x_i \in W$, $R^* \sigma_0 \sigma_1$ iff $\sigma_1 = \sigma_0 \circ x$ for some $x \in W$, and $o^* = \langle o \rangle$.

Given a valuation $v$ on $F$, define a valuation $v^*$ on $F^*$ as $(o, x_1, \cdots, x_m) \in v^*(p_n)$ iff $x_m \in v(P_n)$. When $M = \langle F, v \rangle$ is a model on $F$, let $M^* = \langle F^*, v^* \rangle$.

Define another model $M' = \langle W^*, R', o^*, v^* \rangle$, where $R'(\sigma_0 \circ x)(\sigma_1 \circ y)$ iff $R xy$. The frame $F'' = \langle W^*, R', o^* \rangle$ is to reflect the property of $R$ given by $G$, but it does not necessarily hold. The lemma below is shown by induction.

**Lemma 31.** Let $G$ contain no positive occurrences of equality, $G$ is valid in $F'$. Thus, $F'' \in C$. 

Given $F = \langle X^*, <, v \rangle$, define $*^{-1}(F) = \langle X, *^{-1}(v) \rangle$, where $*^{-1}(v)$ is defined as $x \in (*^{-1}(v))(p)$ iff $\sigma \circ x \in v(p_n)$.

**Observation 32.** $F$ and $F^*$ (so $M$ and $M^*$) are bisimilar via $Zx(o, \cdots, y)$ iff $x = y$. $F$ and $F'$ (so $M$ and $M'$) are also bisimilar via $Zx(o, \cdots, y)$ iff $x = y$. Thus, applying the bisimulation lemma, Let $M = \langle F; v \rangle = \langle W, R, o, v \rangle$ and $F^*$ and $F'$ are as defined above. For each $x \in W$, $M, x \models A$ iff $M^*, \sigma \circ x \models A$ iff $M', \sigma \circ x \models A$.

**Definition 33.** Let $x, y$ be variables and $C$ a predicate variable. $R(x, y, C)$ is adequate for $G$ if

1. $S\omega S \vdash a < b \land a \in T_g \land b \in T_g \rightarrow R(x, y, T_g)$
2. For any $<\text{-convex } C$, i.e., such $C$ that $x \in C \iff (x = \epsilon \lor \epsilon < x)$ and $x \in C \land y < x \rightarrow y \in C$, $G$ is valid in $\langle C, R^C, \epsilon \rangle$, where $R^C = \{\langle x, y \rangle : R(x, y, C)\}$. 
3. $R^T \subseteq g(R')$.

Notice that "$C$ is a $<\text{-convex}" is definable in $S\omega S$.

**Theorem 34.** Let $L$ be a logical system which is complete for a class $C$ of frames defined by $G$, which contains no positive occurrence of equality. Assume $R$ is adequate for $G$. Then, $L$ is decidable.

**Proof.** Let $M = \langle F, v \rangle$ be a model of a formula $A$ on a frame $F = \langle W, R, o \rangle \in C$. We can assume that $M$ is countable, because the language is countable. Also we can assume without loss of generality that $M$ is generated from $o$ and $A$ is true at the root $o$. Since $R$ is adequate, there is an embedding $g$ of $F^*$ into $T = \langle X^*, R^X, \epsilon \rangle$. Define a truth-value assignment $a$ on $g(F)$ as $\sigma \in a(P_n)$ iff $g^{-1}(\sigma) \in v(p_n)$. Let $g(M) = \langle g(F), a \rangle$. 


CLAIM 35. For each formula $B$, $B$ is true at $x$ in $M$ iff $B$ is true at $x^*$ in $M^*$ iff $B$ is true at $g(x)$ in $g(M)$.

The first equivalence is shown in Lemma 32. The second equivalence above is shown by induction; below is the $\square$ case. Assume $\square B$ is not true at $x$ in $M$. Then there is $y$ in $M$ such that $Rxy$ and $B$ is not true at $g(y)$ in $g(M)$. Conversely, assume $\square B$ is not true at $g(x)$ in $g(M)$. Then there is $\sigma$ in $g(M)$ such that $R(g(x), \sigma, g(W))$ and $B$ is not true at $\sigma$ in $g(M)$. By induction hypothesis, $B$ is not true at $g^{-1}(\sigma)$ in $M'$ and $R'xg^{-1}(\sigma)$ by definition of adequacy. By Lemma 32, $B$ is not true at $g^{-1}(\sigma)$ in $M^*$ and $B$ is not true at $g^{-1}(\sigma)$ in $M'$ are equivalent. Thus, $B$ is not true at $g^{-1}(\sigma)$ in $M^*$. Therefore, $\square B$ is true at $x$ in $M$ iff $\square B$ is true at $g(x)$ in $g(M)$ (the end of proof of the claim).

In particular, $A$ is true at $g(o)$ in $g(M)$.

Let $R$ be adequate for $G$ that defines the class $C$ of frames of $\mathcal{L}$. Define a translation $(\ )^\sharp$ from modal formulas to second-order formulas with a free variable $y$ as below:

1. $(p_n)^\sharp(y) = (y \in P_n)$
2. $(A \land B)^\sharp(y) = A^\sharp(y) \land B^\sharp(y)$
3. $(\neg A)^\sharp(y) = \neg A^\sharp(y)$
4. $(\square A)^\sharp(y) = \forall x(R(y, x, T) \rightarrow A^\sharp(x))$

By induction, $B$ is true at $\epsilon$ in $g(M)$ iff $B^\sharp(\epsilon)$ holds in $g(C)$.

CLAIM 36. Suppose $\mathcal{L}$ is complete for $C$ defined by $G$ and $R$ is adequate for $G$. Let $B$ be an arbitrary formula in which occur $p_0, \ldots, p_n$. Then: $\mathcal{L} \vdash B$ iff $\omega \vdash (\forall C, P_i)("C is a convex" \land \bigwedge_{i=0}^{n} P_i \subseteq C) \rightarrow B^\sharp(\epsilon)$. 

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Proof of the claim: Assume \( \mathcal{L} \vdash B \). Then \( B \) is valid in any structure where \( G \) is valid.

Since \( \mathcal{L} \) characterizes \( C \), \( \mathcal{L} \not\vdash B \) implies that there is a generated model \( M \) in \( C \) such that \( B \) is false at the root \( o \) and \( G \) is valid in \( M \). As \( \mathbf{R} \) is adequate for \( G \), \( B \) is false at \( \epsilon \) in \( g(M) \). Thus, by Claim 5.3, \( S\omega S \not\vdash (\forall C, P_i)("C is a convex" \land \bigwedge_{i=0}^{n} P_i \subseteq C) \rightarrow B^\#(\epsilon) \). (the end of proof of the claim).

As the translation \( \# \) is effective and \( S\omega S \) is decidable, in the light of Theorem 28, \( \mathcal{L} \) is decidable. (The end of proof of Theorem 34)

6. Difference operator

Languages \( L(\langle \neq \rangle) \) and \( L(\Diamond, \langle \neq \rangle) \) are defined as in De Rijke [15], [55].

The system \( DL_m \) consists of the system \( K \) for \( \Diamond \) and:

\[
\begin{align*}
(1) & \quad \neg \langle \neq \rangle \neg(p \rightarrow q) \rightarrow (\neg \langle \neq \rangle \neg p \rightarrow \neg \langle \neq \rangle \neg q) \\
(2) & \quad p \rightarrow (\neg \langle \neq \rangle \neg \langle \neq \rangle) p \text{ (symmetry)} \\
(3) & \quad (\langle \neq \rangle \langle \neq \rangle) p \rightarrow (p \lor \langle \neq \rangle p) \text{ (pseudo-transitivity)} \\
(4) & \quad \Diamond p \rightarrow (p \lor \neg \langle \neq \rangle p) \text{ (interaction axiom)} \\
(5) & \quad \text{If } A \text{ is provable, so is } \langle \neq \rangle A.
\end{align*}
\]

Let \( L \) be a logic of \( L(\Diamond, \langle \neq \rangle) \). A set of formulas \( \Gamma \) is \( L\)-nice if it is maximal consistent, and for some propositional letter \( p \), \( p \land \neg \langle \neq \rangle \neg p \in \Gamma \).

**Proposition 37.** A consistent set can be extended to a nice set.

**Proof.** Let \( \Sigma \) be consistent. Pick up a propositional variable \( p^* \) which does not appear in \( \Sigma \). We can do it since we may add a new propositional variable as we want. Let \( \Gamma_0 = \Sigma \cup \{ p^* \land \neg \langle \neq \rangle \neg p^* \} \). Since \( p^* \) is new to \( \Sigma \), \( \Gamma_0 \) is consistent. Enumerate
formulas $A_0, A_1, \cdots$. Construct

$$
\Gamma_{i+1} = \begin{cases} 
\Gamma_i \cup \{A_i\} & \text{if } \Gamma_i \not\vdash \neg A_i \\
\Gamma_i & \text{otherwise}
\end{cases}
$$

and let $\Gamma' = \bigcup_{i<\omega} \Gamma_i$. As in usual Lindenbaum argument, $\Gamma'$ is maximal consistent. Moreover, $\Gamma'$ is nice.

Let $L_{(\not\#), S5}$ be $DL_m + S5$ on $L(\Diamond, (\not\#))$.

**Proposition 38.** Let $\Sigma$ be consistent. Then $\neg(\not\#)(\Sigma) = \{A : \neg(\not\#)\neg A \in \Sigma\}$ is consistent.

**Proof.** Assume $\neg(\not\#)(\Sigma)$ is inconsistent. Then there is $A_0, \cdots, A_n \in \neg(\not\#)(\Sigma)$ such that $\vdash A_0 \land \cdots \land A_n \rightarrow \bot$. As $\neg(\not\#)$ is monotone, $\vdash \neg(\not\#)(A_0 \land \cdots \land A_n) \rightarrow \neg(\not\#)\neg \bot$. But since $\vdash \neg(\not\#)\neg B_0 \land \neg(\not\#)\neg B_1 \leftrightarrow \neg(\not\#)(B_0 \land B_1)$, $\not\#(A_0 \land \cdots \land A_n) \in \Sigma$ thus $\neg(\not\#)\neg \bot \in \Sigma$. It contradicts with $\vdash (\not\#)\top$. □

**Proposition 39.** For each MCS $\Gamma, \Gamma'$, $\{A : \neg(\not\#)\neg A \in \Gamma\} \subseteq \Gamma'$ iff $\{\not\#B : B \in \Gamma'\} \subseteq \Gamma$.

**Proof.** As usual. □

A nice set $\Gamma$ is $\langle \not\# \rangle$-bad in $X$ if there is $\langle \not\# \rangle B \in \Gamma$ such that there is no nice set $\Gamma'$ in $X$ with $B \in \Gamma'$. Let us call a set $X$ of nice sets $\langle \not\# \rangle$-closed if there is no $\langle \not\# \rangle$-bad $\Gamma$ in $X$. A nice set $\Gamma$ is $\Diamond$-bad in $X$ if there is $\Diamond B \in \Gamma$ such that there is no nice set $\Gamma'$ in $X$ with $B \in \Gamma'$ and $C \in \Gamma'$ for every $\Box C \in \Gamma$. Let us call a set $X$ of nice sets $\Diamond$-closed if there is no $\Diamond$-bad $\Gamma$ in $X$.

**Proposition 40.** $\Diamond$-bad set $\Gamma$ in $X$ is $\langle \not\# \rangle$-bad in $X$. 
Proof. By assumption, there is $\Diamond B \in \Gamma$ such that there is no nice set $\Gamma'$ in $X$ with $B \in \Gamma'$ and $C \in \Gamma'$ for every $\Box C \in \Gamma$. By the interaction axiom, $\Diamond B \rightarrow B \lor (\not\neq)B \in \Gamma$, so $B \lor (\not\neq)B \in \Gamma$. But since $\Gamma$ is $\Diamond$-bad, $B \not\in \Gamma$, thus $(\not\neq)B \in \Gamma$. If $\Gamma$ is not $(\not\neq)$-bad, there must be $\Gamma_0$ with $B \in \Gamma_0$, but it contradicts with $\Diamond$-badness of $\Gamma$. □

Corollary 41. A $(\not\neq)$-closed set $X$ of nice sets is $\Diamond$-closed.

Proposition 42 (Killing lemma). Let $\Gamma$ be $(\not\neq)$-bad in $X$. Then there is a set $X'$ of nice sets with $X \subseteq X'$ such that $\Gamma$ is not $(\not\neq)$-bad in $X'$.

Proof. Let $\Gamma_0 = \{B : (\not\neq)B \in \Gamma\}$. As $\Gamma$ is consistent, so is $\Gamma_0$. Extend $\Gamma_0$ to a nice set $\Delta$, and let $X' = X \cup \{\Delta\}$. Then $\Gamma$ is not $(\not\neq)$-bad in this $X'$. □

Proposition 43. $L_{(\not\neq),S5}$ is strongly complete.

Proof. Soundness is routine. Let $\Sigma \not\vdash_{L_{(\not\neq),S5}} A$. Extend $\Sigma \cup \{\neg A\}$ to an $L_{(\not\neq),S5}$-nice set $\Gamma_0$. Let us construct a set $W$ of nice sets with no $(\not\neq)$-bad one in $W$. Let $W_0 = \{\Gamma_0\}$, and consider $W_i$. If there is any bad $\Gamma \in W_i$, extend $W_i$ to $W_{i+1}$ to kill badness of $\Gamma$. Define $W^* = \cup_{i<\omega} W_i$. Assume there is any bad $\Gamma$. But there should be $n$ with $\Gamma = \Gamma_n$, and its badness is killed at the stage $n + 1$. Thus, $W^*$ is $(\not\neq)$-closed, thus $\Diamond$-closed.

Define: $R'_{(\not\neq)} \Gamma \Gamma'$ iff $\{A : \neg (\not\neq)\neg A \in \Gamma\} \subseteq \Gamma'$. It is easy to show the relation is symmetric and pseudo-transitive. We have to make it irreflexive. A construction is as follows. If there are any $\Gamma \in W^*$ and $(\not\neq)B \in \Gamma$ such that the only nice set in $W^*$ containing $B$ is $\Gamma$, duplicate it to $\Delta, \Delta'$ and revise $R_{(\not\neq)}$. The resulting $W$ is $(\not\neq)$-closed.

Need to check: for any $\Gamma, \Gamma' \in W$, $R_{(\not\neq)} \Gamma \Gamma'$ iff $\Gamma \neq \Gamma'$. From the right to left is just constructed. Conversely: First, $R_{(\not\neq)} \Gamma \Gamma'$ is symmetric, pseudo-transitive, and irreflexive. If it is not inequality, there are $\Gamma_0, \Gamma_1$ such that for every $\Delta \in W, \neg R_{(\not\neq)} \Gamma_0 \Delta$
and \( \neg R(\neq) \Delta_1 \). But it contradicts with the fact from the construction of \( W \) that for every \( \Delta_0, \Delta_1 \in W \) such that \( R(\neq) \Delta_0 \Delta_1 \) or \( \Delta_0 = \Delta_1 \).

Define a structure \( \langle W^*, R_\neq, \neq, v \rangle \) with \( W^* \) defined as the above, \( R_\neq \Gamma \Gamma' \) iff \( \{ A : \Box A \in \Gamma \} \subseteq \Gamma' \), and \( \Gamma \in v(p) \) iff \( p \in \Gamma \). It is routine to show that \( R_\neq \) is as desired.

Truth lemma (sketch): Check only the \( \langle \neq \rangle \) clause: \( \langle \neq \rangle B \in \Gamma \) iff there is \( \Gamma' \neq \Gamma \) such that \( B \in \Gamma' \). From the left to the right: immediately from \( \langle \neq \rangle \)-closedness. From the right to the left: Assume there is such \( \Gamma' \). Then \( \langle \neq \rangle B \in \Gamma \).

7. Neighborhood semantics: basics

Logics of neighborhood frames are extensively discussed in Segerberg [62].

7.1. Definitions. The language in the section has a modal operator \( \star \).

A neighborhood frame is \( F = \langle W, \nu \rangle \) where \( W \) is a non-empty set and \( \nu \) is a function which assigns a set of subsets of \( W \) to \( w \in W \). A valuation on a neighborhood frame \( F \) is \( \nu : \text{Prop} \to \wp W \). A neighborhood model is \( M = \langle F, v \rangle \) where \( F \) is a neighborhood frame and \( v \) is a valuation on \( F \).

Atomic and boolean sentences are interpreted as usual. The truth condition of modal sentences is: \( M, w \models \star A \) iff there is \( S \in \nu(w) \) such that \( x \in S \) iff \( M, x \models A \).

7.2. Classical modal logics. A modal logic is called classical if it contains all theorems of PC and it is closed under the rule:

\[
\frac{A \leftrightarrow B}{\star A \leftrightarrow \star B}
\]

The minimal classical modal logic is axiomatized with PC theorems and the above rule.

7.3. Construction of a canonical model. Completeness of the minimal classical logic is shown by a construction of a canonical model.
Let $U_c$ be the class of maximal consistent sets. For each formula, let $|A| = \{x \in U_c : A \in x\}$. Define $\nu$ to be: $\nu(x) = \{y \subseteq U_c : \text{for some formula } A, \star A \in x \text{ and } |A| = y\}$. Define the canonical valuation $v_c$ as $v_c(p) = |p|$. A canonical model is $M_c = \langle U_c, \nu_c, v_c \rangle$.

**Proposition 44.** A is a theorem of the basic classical modal logic $L$ iff $A$ has a neighborhood model.

**Proof.** Soundness is trivial.

Completeness: Let $A$ be a consistent formula. There is a maximal consistent extension of $\{A\}$. Let us call it $u_A$. Consider a canonical model. Then $u_A$ is its point and $M_c, u_A \models A$. $\square$

### 7.4. Neighborhood semantics and relational semantics.

A class of neighborhood semantics can be translated to relational semantics.

**Definition 45.** Let $F = \langle W, \nu \rangle$ be a neighborhood frame. $x \in W$ is normal iff $\nu(x)$ is a filter. $x \in W$ is singular iff $\nu(x) = \emptyset$. $F$ is regular iff every $x \in W$ is either normal or singular. $F$ is normal iff every $x \in W$ is normal. $F$ is singular iff every $x \in W$ is singular.

Let $F = \langle W, \nu \rangle$ be regular. The alternative relation induced by $F$ if $Rxy$ iff $x$ is normal and $y \in \bigcap \nu(x)$.

$\langle W, R, Q \rangle$ is a regular relational frame where $R$ is an alternative relation induced by a regular neighborhood frame and $Q \subseteq$ is the set of singular points. $\langle W, R, Q \rangle$ is a normal relational frame if $Q = \emptyset$.

A modal sentence $\star A$ is interpreted in a regular relational frame as:

$M, x \models \star A$ iff $x \not\in Q$, and for all $y(Rxy \Rightarrow M, y \models A)$
Proposition 46. For every normal $x$, $F, v, x \models A$ iff $W, R, v, x \models A$. In particular, $F, v, x \models \star A$ iff $W, R, v, x \models \star A$ with the standard truth condition of relational possible world semantics, i.e., $W, R, v, x \models \star A$ iff for every $y$, if $Rxy$ then $W, R, v, y \models A$.

Proof. By induction on $A$. Suppose $F, v, x \models \star A$, i.e., there is $S \in \nu(w)$ such that $x \in S$ iff $M, x \models A$. Suppose $Rxy$. Then $x$ is normal and $y \in \bigcap \nu(x)$. As $y \in S$, by inductive hypothesis $W, R, v, y \models A$.

Conversely, suppose $F, v, x \not\models \star A$. It follows that for every $S \in \nu(w)$, there is $z \in S$ such that $M, z \not\models A$ or there is $z \notin S$ such that $M, z \models A$. As $x$ is normal, for every $S \in \nu(w)$, there is $z \in S$ such that $M, z \not\models A$, Thus, there is $y$ such that $Rxy$ and $M, y \not\models A$. \qed

Corollary 47. Validity in regular (normal) neighborhood frames coincides with validity in regular (normal) relational frames.

7.5. Syntactical characterization of regular and normal logics. Regular and normal logics can be syntactically characterized.

\[
K \ (\star A \land \star B) \to \star(A \land B) \\
\neg A \star(A \land B) \to (\star A \land \star B) \\
\neg (\star \top)
\]

Proposition 48. A modal logic $L$ is regular iff both $K$ and $A$ are provable in $L$. A modal logic $L$ is normal if $K$, $A$, and $N$ are provable in $L$.

7.6. Characterizing neighborhood frames. Like well-known results for normal logics, there are characterizing axioms for some classes of neighborhood frames.
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⋆\top \quad \text{normal frames}

⋆p \quad \text{singular frames (i.e., for every } x \in U, \nu(x) = \emptyset) \quad ⋆p \to p(T) \quad \text{for every } x \in U \text{ if } \nu(x) \neq \emptyset \text{ then } x \in \bigcap \nu(x)

7.6.1. Neighborhood semantics and relational semantics. A class of neighborhood semantics can be translated to relational semantics.

**Definition 49.** Let \( F = \langle W, \nu \rangle \) be a neighborhood frame. \( x \in W \) is normal iff \( \nu(x) \) is a filter. \( x \in W \) is singular iff \( \nu(x) = \emptyset \). \( F \) is regular iff every \( x \in W \) is either normal or singular. \( F \) is normal iff every \( x \in W \) is normal. \( F \) is singular iff every \( x \in W \) is singular.

Let \( F = \langle W, \nu \rangle \) be regular. The alternative relation induced by \( F \) if \( Rxy \) iff \( x \) is normal and \( y \in \bigcap \nu(x) \).

\( \langle W, R, Q \rangle \) is a regular relational frame where \( R \) is an alternative relation induced by a regular neighborhood frame and \( Q \subseteq \) is the set of singular points. \( \langle W, R, Q \rangle \) is a normal relational frame if \( Q = \emptyset \).

**Proposition 50.** For every normal \( x \), \( F, v, x \models A \) iff \( W, R, v, x \models A \). In particular, \( F, v, x \models \Box A \) iff \( W, R, v, x \models \Box A \) with the standard truth condition of relational possible world semantics, i.e., \( W, R, v, x \models \Box A \) iff for every \( y \), if \( Rxy \) then \( W, R, v, y \models A \).

**Proof.** By induction on \( A \). Suppose \( F, v, x \models \Box A \), i.e., there is \( S \in \nu(w) \) such that \( x \in S \) iff \( M, x \models A \). Suppose \( Rxy \). Then \( x \) is normal and \( y \in \bigcap \nu(x) \). As \( y \in S \), by inductive hypothesis \( W, R, v, y \models A \).

Conversely, suppose \( F, v, x \not\models \Box A \). It follows that for every \( S \in \nu(w) \), there is \( z \in S \) such that \( M, z \not\models A \) or there is \( z \notin S \) such that \( M, z \models A \). As \( x \) is normal, for every \( S \in \nu(w) \), there is \( z \in S \) such that \( M, z \not\models A \). Thus, there is \( y \) such that \( Rxy \) and \( M, y \not\models A \). \( \square \)
Corollary 51. Validity in regular (normal) neighborhood frames coincides with validity in regular (normal) relational frames.
CHAPTER 6

Logics on partitions

Information is context-dependent. Moreover, our life is more complicated: Ambiguity of a message suggests that even in a single situation more than one context can exist.

Taking it for granted that a partition of possible worlds serves as at least a partial representation of a context, the idea of multi-contextual situation motivates the proposal of partitionistic structures in this paper.

The chapter aims to investigate logics arising from partitionistic structures. In addition to completeness and the finite model property of the basic partition logic and some characterization results, it is shown that the class of p-frames with linearly ordered partitions cannot be characterized.

1. Language and semantics

The modal language here has modal operators $[\forall]$ and $\nabla$.

A p-frame is $(W, \Pi)$, where $W$ is an non-empty set, and $\Pi$ is a set of partitions on $W$, i.e. each $P \in \Pi$ is a partition of $W$. A valuation $v$ on p-frame $F = (W, \Pi)$ is a function which assigns a subset of $W$ to a propositional letter. A p-model on a p-frame $F = (W, \Pi)$ is $(W, \Pi, v)$, where $v$ is a valuation on $F$.

Let $[x]_P$ stand for the member $U \in P \in \Pi$ such that $x \in U$.

The truth condition for $\nabla$ is: $M, x \models \nabla A$ iff there is $P \in \Pi$ such that $[x]_P$ is the truth set of $A$ in $M$; and the truth condition for $[\forall]$ is: $M, x \models [\forall]A$ iff $M, w \models A$.

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for every world \( w \in W \). When \( \nabla A \) holds at \( x \) in \( M \), a witness for \( \nabla A \) at \( x \) in \( M \) is \( P \in \Pi \) such that \( [x]_P \in P \). It may not be unique.

Substitution to propositional variables works as usual.

2. Comparison with neighborhood semantics

Segerberg [62] investigates neighborhood semantics of modal logic. A neighborhood frame \( F \) is a pair \( \langle U, \nu \rangle \), where \( U \) is a set and \( \nu \) is a function which assigns to \( x \in U \) a set of subsets of \( U \); a truth value assignment \( v \) on \( F \) is a function from propositional variables to the powerset of \( U \); a neighborhood model on \( F \) is a pair \( \langle F, v \rangle \) where \( v \) is a truth value assignment on \( F \); and the notion is recursively defined as usual; in particular, \( \star A \) is true at \( x \in U \) iff there is \( s \in \nu(x) \) such that \( s \) is the truth set of \( A \).

A neighborhood frame \( F \) is reflexive if for every \( x \in U \) if \( \nu(x) \neq \emptyset \) then \( x \in \bigcap \nu(x) \). It is easy to observe that \( F \) is reflexive iff the scheme \( \star p \to p \) is valid in \( F \). In fact, for each p-frame (or each p-model), a reflexive neighborhood frame (model) can be defined as follows. Given a p-frame \( F_p = \langle W, \Pi \rangle \), define a neighborhood frame \( N(F_p) = \langle W, \nu \rangle \) where the neighborhood set \( \nu(x) \) of \( x \) is defined as \( \nu(x) = \{[x]_P : P \in \Pi \} \). Take \( v \) on the p-frame as a valuation on the corresponding neighborhood frame. Then:

\[
\text{Proposition 52. } F_p, v, x \models A \text{ iff } N(F_p), v, x \models A.
\]

In particular, when \( \Pi \neq \emptyset \), \( \nu(x) \neq \emptyset \) for each \( x \).

There is no reverse translation; for the class of all p-frames is a subclass of the class of all reflexive neighborhood frames, with any neighborhood sets for each point are nested. Thus a stronger logic is expected; is there a strictly stronger logic? The answer is yes.
3. Basic partition logic: axiomatic system \( L([\forall], \nabla) \)

(1) PC
(2) S5 for \([\forall]\)
(3) \([\forall](p \rightarrow \nabla p) \lor [\forall](p \rightarrow \neg \nabla p)\)  
\hspace{1cm} (Partition axiom)
(4) \(\nabla p \rightarrow p\) (Reflexivity axiom)

Soundness is easy. The following are deducible from the above:

- \([\forall] \nabla \top \lor [\forall] \neg \nabla \top\)
- \(\langle \exists \rangle \nabla A \rightarrow [\forall](\nabla A \leftrightarrow A)\)
- \(\langle \exists \rangle (A \land \neg \nabla A) \rightarrow [\forall] \neg \nabla A\)
- \([\forall](p \rightarrow q) \rightarrow ([\forall](\nabla p \rightarrow \nabla q) \lor [\forall](\nabla p \rightarrow \neg \nabla q))\)
- \([\forall](A \leftrightarrow B) \rightarrow [\forall](\nabla A \leftrightarrow \nabla B)\)
- \((\exists)\nabla B \land (\exists)\nabla \neg B \rightarrow [\forall](\nabla B \lor \nabla \neg B)\)

If \(X\) is a finite set of formulas, \(\bigwedge X\) is the conjunction of all formulas in \(X\), and \(\bigvee X\) is the disjunction of them. (Let \(\bigwedge \emptyset = \top\) and \(\bigvee \emptyset = \bot\) conventionally.)

4. Completeness and the finite model property

Completeness and the finite model property are shown simultaneously by the following construction of a finite model of a given consistent formula, where each point is a formula.

Let \(A\) be a consistent formula. Pick a maximal consistent set \(X_A\) such that \(A \in X_A\). Let \(\Sigma\) be the set of maximal consistent sets \(Y\) such that \([\forall]^-X_A = \{B : [\forall]B \in X_A\} \subseteq Y\).
Let \( Prop_A \) be the set of propositional variables occurring in \( A \). For the set \( Sub(A) \) of all subformulas of \( A \), define:

\[
\begin{align*}
\Gamma_0 &= \text{Sub}(A) \cup \{ \neg B : B \in \text{Sub}(A) \} \\
\Gamma_1 &= \Gamma_1 \cup \{ \nabla B : B \in \Gamma_0 \} \\
\Gamma_2 &= \Gamma_2 \cup \{ \neg [\forall] \neg B : B \in \Gamma_1 \} \\
\Gamma_A &= \bigcup_{B \in \Gamma_2} \text{Sub}(B)
\end{align*}
\]

Also define for each \( Y \in \Sigma \),

\[
\begin{align*}
D(Y) &= \Gamma_A \cap Y \\
C(Y) &= \bigwedge D(Y)
\end{align*}
\]

Let \( \Omega = \{ B \in \text{Sub}(A) : \neg [\forall] \neg \nabla B \in X_A \} \).

**Proposition 53.** Let \( B \in \Omega \) and \( B' \in \text{Sub}(A) \). If \( [\forall](B \leftrightarrow B') \in X_A \) then \( B' \in \Omega \).

**Proof.** It holds because \( [\forall](A \leftrightarrow B) \rightarrow [\forall](\nabla A \leftrightarrow \nabla B) \) is a theorem. \( \square \)

Let \( U_0 = \{ C(Y) : Y \in \Sigma \} \). Since \( \Gamma_A \) is finite, so is \( U_0 \).

**Observation 54.** From the definitions,

- For any \( C \in U_0 \), \( C \not \vdash \bot \).
- For any \( C \in U_0 \), \( \exists C \in X_A \).
- There is a unique \( C \) among \( C_i \)'s such that \( C \in X_A \).
- For any \( B \in \text{Sub}(A) \), if \( \exists B \in X_A \), there exists \( C \in U_0 \) such that \( C \vdash B \).
- For any \( C \in U_0 \), and for any \( B \) on \( Prop_A \), if \( C \vdash [\forall]B \), then \( [\forall]B \in X_A \).
- If \( C \neq C' \), \( \vdash C \land C' \leftrightarrow \bot \).
- \( C \vdash \exists C' \) for any \( C, C' \in U_0 \).
- Let \( E_A = \bigwedge (\Gamma_A \cap [\forall]^{-} X_A) \). Then
- \( C \vdash E_A \) for each \( C \in U_0 \).
- \( \vdash \bigvee_{C \in U_0} C \leftrightarrow E_A \).

Classify \( \Omega \) as follows:

\[
\begin{align*}
O^T &= \{ B \in \Omega : [\forall]\nabla B \in X_A \} \\
O^+ &= \{ B \in \Omega : [\exists]\nabla \neg B \in X_A \} \\
O^- &= \{ B \in \Omega : -[\forall]\nabla B \land [\forall]\neg \nabla \neg B \in X_A \}
\end{align*}
\]

These sets are mutually disjoint and exhaust \( \Omega \).

Intuitively, formulas in \( O^+ \) and \( O^T \) behave well, while those in \( O^- \) require taming.

**Proposition 55.** Properties of formulas in \( O^+ \) and \( O^T \): For each \( B \in \Omega \),

- \( B \in O^T \) iff for each \( C \in U_0 \), \( C \vdash [\forall]\nabla B \).
- Suppose \( B \in O^T \). Then \( [\forall]B \in X_A \).
- If \( B \in O^+ \), then there exists \( C \in U_0 \) such that \( C \vdash B \), there exists \( C' \) such that \( C' \vdash \neg B \). Moreover, for any \( C \in U_0 \), \( C \vdash B \) or \( C \vdash \neg B \).

**Proof.** Note that all formulas in this argument are on \( Prop_A \), and that in particular every \( p \in Prop_A \) occurs in \( C \).

- Right to left is obvious. Suppose \( B \in O^T \). Then as \( [\forall]\nabla B \in X_A \), for every \( C \in U_0 \), \( C \vdash [\forall]\nabla B \).
- By theorem \( (\exists)\nabla B \rightarrow [\forall](\nabla B \leftrightarrow B) \).
- Suppose \( B \in O^+ \). Then \( (\exists)\nabla B \in X_A \) and \( (\exists)\nabla \neg B \in X_A \). Thus, there exists \( C \in U_0 \) such that \( C \vdash B \) and there exists \( C' \in U_0 \) such that \( C' \vdash \neg B \).

As \( ((\exists)\nabla B \land (\exists)\nabla \neg B) \rightarrow [\forall](\nabla B \lor \nabla \neg B) \) is a theorem, \( [\forall](\nabla B \lor \nabla \neg B) \in X_A \), so \( C_i \vdash \nabla B \lor \nabla \neg B \) for each \( C_i \). Moreover, since \( (\exists)\nabla A \rightarrow [\forall](\nabla A \leftrightarrow A) \) is a theorem, \( C \vdash \nabla B \leftrightarrow B \) and \( C \vdash \nabla \neg B \leftrightarrow \neg B \) for each \( C \) for every \( C \in U_0 \).
– Suppose \( C \not \vdash B \). Then, \( C \not \vdash \neg B \). Thus \( C \vdash \neg \neg B \) which entails \( C \vdash B \).

– Suppose \( C \not \vdash \neg B \). Then, \( C \not \vdash \neg B \). Thus \( C \vdash \neg \neg B \) which entails \( C \vdash B \).

Thus, for each \( C \), \( C \vdash B \) or \( C \vdash \neg B \).

\( \square \)

Let \( U_f = U_0 \times \{0\} \cup U_0 \times \{1\} \). Introduce partitions on \( U_f \) for each \( B \in \Omega_A \) as follows:

- \( B \in O^\top \): \( P_B = \{U_f\} \).
- \( B \in O^+ \) \( P_B = \{(C, j) : C \vdash B, j = 0.1\} \)
- \( B \in O^- \):

  \[
  P_B = \{(C, j) : C \vdash B, j = 0.1\}, \{(C, 0) : \vdash C \rightarrow B\}, \{(C, 1) : \vdash C \rightarrow B\}
  \]

Let \( \Pi_f = \{P_B : B \in \Omega\} \).

**Proposition 56.** Let \( B \in \text{Sub}(A) \). There is \( P \in \Pi_f \) such that \( \{(C, j) : C \vdash B, j = 0.1\} \in P \) iff \( B \in \Omega \).

**Proof.** Right to left is by definition. Suppose there is \( P \in \Pi_f \) such that \( \{(C, j) : C \vdash B, j = 0.1\} \in P \). Then, there is \( B' \in \Omega \) such that There is \( P \in \Pi_f \) such that \( \{(C, j) : C \vdash B', j = 0.1\} \in P \). By Proposition 53, \( B \in \Omega \).

Define \( v = f : \text{Prop} \rightarrow \wp U_f \) as: \( v_f(p) = \{(C, j) : C \vdash p\} \) if \( p \in \text{Prop}_A \), and \( v_f(p) = \emptyset \) otherwise.

Define \( M_f = \langle U_f, \Pi_f, v_f \rangle \). Then \( M_f \) is a p-model.

**Lemma 57.** Let \( B \in \text{Sub}(A) \). Then \( M_f, (C, j) \models B \) iff \( C \vdash B \).

**Proof.** By induction on \( B \).

- \( B \) is a propositional variable. \( M_f, (C, j) \models B \) iff \( C \vdash p \).

- A standard argument works when \( B = B_0 \land B_1 \) or \( B = \neg B_0 \).
6. LOGICS ON PARTITIONS

• When \( B = [\forall B_0] \). \( M_f, \langle C, j \rangle \models [\forall B_0] \) iff for every \( \langle C', j' \rangle \) \( M_f, \langle C', j' \rangle \models B_0 \) iff for every \( \langle C', j' \rangle \) \( C' \vdash B_0 \) iff \( \bigvee_{C' \in U_0} C' \vdash B_0 \) iff \( E_A \vdash B_0 \) iff \( \exists \forall B_0 \). 

• \( B = \nabla B_0 \). Suppose \( C \vdash \nabla B_0 \). Then \( (\exists) \nabla B_0 \in X_A \), so \( B_0 \in \Omega \). By definition of \( \Pi_f \), for any member of \( \Omega \), there is \( P \in \Pi_f \) such that \( \langle C, j \rangle \in \{ \langle C', j' \rangle : C' \vdash B_0 \} \in P \). By induction hypothesis, it is equivalent to that there is \( P \in \Pi_f \) such that \( \langle C, j \rangle \in \{ \langle C', j' \rangle : M_f, \langle C', j' \rangle \models B_0 \} \in P \), which is equivalent to \( M_f, \langle C, j \rangle \models \nabla B_0 \).

Conversely, suppose \( M_f, \langle C, j \rangle \models \nabla B_0 \). It is equivalent to that there is \( P \in \Pi_f \) such that \( \langle C, j \rangle \in \{ \langle C', j' \rangle : M_f, \langle C', j' \rangle \models B_0 \} \in P \), which in turn is equivalent by induction hypothesis to that there is \( P \in \Pi_f \) such that \( \langle C, j \rangle \in \{ \langle C', j' \rangle : C' \models B_0 \} \in P \).

For such a set is in \( P \), \( B_0 \in \Omega \) must hold by Proposition 56. 

- \( B_0 \in O^\top \): Then by Proposition 55 for any \( C \vdash [\forall] B_0 \), and moreover, as \( (\exists) \nabla B_0 \rightarrow [\forall] (B_0 \leftrightarrow \nabla B_0) \) is a theorem, \( C \vdash \nabla B_0 \).

- \( B_0 \in O^+ \): Then there exists \( C \) such that \( C \vdash B \), there exists \( C \) such that \( C \vdash \neg B \), and for any \( C \), \( C \vdash B \) or \( C \vdash \neg B \). As \( C \vdash B_0 \), and \( [\forall] (B_0 \leftrightarrow \nabla B_0) \), \( C \vdash \nabla B_0 \).

- \( B_0 \in O^- \): \( C \vdash B \) and \( [\forall] (B_0 \leftrightarrow \nabla B_0) \), \( C \vdash \nabla B_0 \).

Thus, in any cases, \( C \vdash \nabla B_0 \).

Therefore, Then \( M_f, \langle C, j \rangle \models B \) iff \( C \vdash B \). \( \square \)

From the argument above,

**Theorem 58.** The class of all p-frames is characterized by the system of basic partition logic.
5. Regular partition logics

The notions of regular and normal p-frames are similar to Segerberg [62]'s on neighborhood frames.

For \( F = \langle W, \Pi \rangle \) and \( x \in W \), let \( \Pi(x) = \{ U \subseteq W : \text{ there is } P \in \Pi \text{ such that } U \in P \} \). A p-frame \( F = \langle W, \Pi \rangle \) is regular iff \( \Pi \) gives each point a filter, or for each \( x \in W \), if \( \Pi(x) \neq \emptyset \) then for all \( U, V \subseteq W \), \( U, V \in \Pi(x) \) iff \( U \cap V \in \Pi(x) \).

The logic of the class of all regular p-frames can be axiomatized by:

- Axioms and rules of the basic partition logic
- \( \Box(p \land q) \rightarrow \Box p \)
- \( \Box p \land \Box q \rightarrow \Box (p \land q) \)
- \( \Box \top \rightarrow (\Box p \leftrightarrow p) \)

The following are deducible:

- \( \vdash A \rightarrow B \) / \( \vdash \Box A \rightarrow \Box B \)
- \( \Box A \rightarrow \Box \top \)
- \( \neg \Box A \lor (\Box A \leftrightarrow A) \)

There are two nice superlogics of the regular partition logic. First, the logic of the class of all p-frames with empty \( \Pi \). \( \neg \Box p \) characterizes the class\(^1\).

Another special case is the logic of the class of normal p-frames, i.e. p-frames where \( \Pi \) gives each point a non-empty filter. It is axiomatizable with

- Axioms and rules of the regular partition logic
- \( \Box \top \)

\( \Box A \leftrightarrow A \) is deducible, i.e. the logic is so-called \( \text{Triv}^2 \).

---

\(^1\)Segerberg considers a similar case.

\(^2\)Hughes-Cresswell [37, p. 65].
Those results are observed by considering a relational frame associated to such a structure. As seen before, for any p-frame $F = \langle W, \Pi \rangle$, there is a neighborhood frame $F_n = \langle W, \nu \rangle$ where $\nu(x) = \Pi(x)$. In particular, when $F$ is a regular p-frame, $F_n$ is regular in the sense of neighborhood semantics.

Let $F = \langle W, \nu \rangle$ be regular. The alternative relation induced by $F$ is $R$ defined as: $Rxy$ iff $x$ is normal\(^3\) and $y \in \bigcap \nu(x)$. Moreover, a relational counterpart of a regular p-frame $F = \langle W, \nu \rangle$ is a relational frame $F_r = \langle W, R, Q \rangle$, where $R$ is the alternative relation induced by $F$ and $Q$ is the set of singular points, or $Q = \{x : \nu(x) = \emptyset\}$.

In the case of p-frames, if $\Pi = \emptyset$ then the corresponding $Q = W$; and if $\Pi \neq \emptyset$, then $Q = \emptyset$. A modal sentence $\star A$ is interpreted in a regular relational frame as:

$M, x \models \star A$ iff $x \notin Q$, and for all $y (Rxy \Rightarrow M, y \models A)$

**Proposition 59.** Let $R$ be the alternative relation induced by the neighborhood frame $F$ generated from a p-frame $F = \langle W, \Pi \rangle$. Then either $R = \emptyset$, or $Rxy$ implies $x = y$.

**Proof.** $\Pi = \emptyset$ implies $Q = W$, which is equivalent to there is no normal point in $W$. Therefore $Rxy = \emptyset$.

Suppose $\Pi \neq \emptyset$. Then, every $x \in W$ is normal. Take an arbitrary $P \in \Pi$ and distinct $x, y \in W$. Since for every superset $S$ of $[x]_P$ there is $P' \in \Pi$ such that $S \in P'$. In particular, $W \setminus \{y\} \in P'$. For $P'$ to be a partition, $\{y\} \in P'$ must hold. Thus, for every $x \in W$, $\{x\} \in \nu(x)$. Suppose $Rxy$. By definition, $y \in \bigcap \nu(x)$; in particular $y \in \{x\}$. Therefore, $x = y$.\(^\Box\)

\(^3\)I.e. $\nu(x)$ is a non-empty filter.
Hence, the class of relational counterparts of neighborhood frames generated from empty p-frames $F = \langle W, \emptyset \rangle$ is the class of singular frames. The logic is characterized with $\neg \nabla p$.

On the other hand, the class of relational counterparts of neighborhood frames generated from non-empty p-frames is the class of normal frames with the frame condition $Rxy$ implies $x = y$. The logic is thus $\text{Triv}$.

6. Singleton partition logic

$$(\nabla p \land \nabla q) \rightarrow [\forall](p \rightarrow q)$$ characterizes the class of frames where $\Pi$ is a singleton.

Let $[\text{unique}] \cdot$ stand for $\nabla$ with the additional axiom in arguments below.

6.1. Size of the partition. The following are further additional correspondence axioms with $[\text{unique}] \cdot$.

- At-least-$n$-equivalence-classes;

$$\bigwedge_{0 \leq i \leq n-1} \langle \exists \rangle [\text{unique}] \cdot\langle (\bigwedge_{0 \leq j \leq n-1} \neg A_j) \land A_i \rangle \rightarrow \langle \exists \rangle \bigwedge_{0 \leq i \leq n-1} \neg A_i$$

- Exactly-1-equivalence-class: $[\text{unique}] \cdot \top$

- Not-2-equivalence-classes (either 1 or more than 2):

$$[\text{unique}] \cdot p \rightarrow [\forall] \neg [\text{unique}] \cdot \neg p$$

6.2. Boolean and all-and-only modality. There are at least two topics related to the singleton partition logic: Boolean modalities and the difference operator.

Let $M = \langle W, R, v \rangle$ be a relational model. $[\text{ext}]$ is interpreted with the truth condition: $M, x \models [\text{ext}]A$ iff for all $y$, if NOT $Rxy$ then $M, x \models A$. Humberstone [38] gives an axiomatic system of the logic of valid sentences in the class of all complementary frames with a schema:
(1) $K$ for $\Box$ and $[\text{ext}]$

(2) Let $S$ be a sequence of $\Box$ and $[\text{ext}]$, and $W$ be the sequence of modalities where each $*$ in $S$ is substituted by its dual. Then,

\[(6) \quad W(\Box p \land [\text{ext}] q) \rightarrow S(p \lor q)\]

$\nabla A$ in singleton frames behaves as $\Box A \land [\text{ext}] A$ with the underlying relational S5 frames.

Goranko [23] investigates logics of various classes of complementary frames further to obtain correspondence results; some cases require interaction axioms between $\Box$ and $[\text{ext}]$ in addition to characteristic axioms in the standard sense.

A more general treatment has been pursued by a Bulgarian group as Boolean modal logic. Fix a collection $C$ of binary relations on $W$, and a boolean algebra$^4$ is obtained on the basis of $C$: i.e. $B_C$ contains $\emptyset, W \times W$ and members of $C$ and closed under $\cap, \cup,$ and $\sim$. Then we can consider a pair $M = (W, B_C)$, and investigate logics on such structures. In connection to Boolean modal logics, a “window operator” is defined for any relation symbol. In other words, for $[\text{ext}]$ interpreted with $R$, $\| A \|$ is defined as $[\text{ext}] \neg A$. For more details, see Blackburn et al. [8, pp.424-425].

6.3. Difference modality. Chapter 5 Section 6 presents the difference modality: $M, x \models (\neq) A$ iff for every $y \neq x$, $M, y \models A$. “$\cdots$ holds only at $x$” can be defined as $A \land (\neq) \neg A$. $\nabla A$ in singleton frames behaves similarly to “only”, in the sense it claims that the proposition holds all and only in the piece of the partition.

$^4$Relaxing it to a regular algebra gains propositional dynamic logic.
7. Less than or equal to $n$ partitions

The result is extended to the case of the class of p-frames with less than or equal to $n$ partitions. The characterizing axiom is:

$$\bigwedge_{1 \leq i \leq n} p_i \land p_{n+1} \rightarrow \bigvee_{1 \leq i \neq j \leq n} (p_i \rightarrow p_j) \lor \bigvee_{1 \leq i \leq n} (p_i \rightarrow p_{n+1})$$

8. Linear order on partitions

8.1. Linear p-frames. Let $W$ be a set, and $\Pi$ be a set of partitions of $W$. Define $\leq_\Pi$ on $\Pi$ as: $P \leq_\Pi P'$ iff, for any $U \in P$, there is $U' \in P'$ such that $U \subseteq U'$. $\leq_\Pi$ is a partial order.

A linear p-frame is a p-frame with $\Pi$ is linearly ordered by $\leq_\Pi$. A linear p-model is a p-model on a linear p-frame.

Consider a system $S$ which has the following axiom in addition to the axioms and rules of the basic partition logic:

- $(\nabla p \land \nabla q) \rightarrow ([\forall](p \rightarrow q) \lor [\forall](q \rightarrow p))$ (Nesting)

8.2. Impossibility to axiomatize the class of linear p-frames. $S$ is sound in the class of all linear p-frames. Moreover, to be shown below, any $S$-consistent formula has a linear p-model. Nevertheless, $S$ is sound in a frame $F = \langle W, \Pi \rangle$ where $W = \{a, b, c\}$, $\Pi = \{\{a\}, \{b, c\}\}, \{\{a, b\}, \{c\}\}$. Clearly $F$ is not a linear p-frame. Thus, the class of all linear p-frames is a proper subclass of that of all $S$-frames. It means that the class of all linear p-frames cannot be characterized; for, if there were an additional characterizing formula of linear p-frames, it must be falsified in the class of $S$-frames, so that it cannot be $S$-theorem, which is absurd.
8.3. Construction of a linear p-frame. Now, a linear p-model of a given $S$-consistent formula $A$ is to be constructed. Some notions and properties must be prepared before the construction.

Let $X_A$, $\Gamma_A$, $\Omega$, and $U_0$ be defined as before. Let $T = \{ V \subseteq U_0 : V \neq \emptyset \text{ and there is } B \in \Omega \text{ such that } V = \{ C \in U_0 : C \vdash B \} \}$. Then for each non-empty $X$, $X \in T$ iff there is $B \in \text{Sub}(A)$ such that $\{ C \in U_0 : C \vdash \exists B \} = X$.

**Proposition 60.** If $\Omega \neq \emptyset$, then $T \neq \emptyset$.

**Proof.**Suppose $\Omega \neq \emptyset$. Then there is $B \in \text{Sub}(A)$ such that $\langle \exists \rangle \exists B \in X_A$. It implies that there is $C \in U_0$ such that $C \vdash \exists B$, and thus $C \vdash B$. \hfill \Box

**Proposition 61.** For all $V, V' \in T$, $V \cap V' \neq \emptyset$ then $V \subseteq V'$ or $V' \subseteq V$.

**Proof.**Suppose $V \cap V' \neq \emptyset$, $V = \{ C \in U_0 : C \vdash B \}$, and $V = \{ C \in U_0 : C \vdash B' \}$. Then there is $C \in U_0$ such that $C \vdash B$ and $C \vdash B'$. Since $\langle \exists \rangle \exists B \in X_A$ and $\langle \exists \rangle \exists B' \in X_A$, $[\forall](\exists B' \rightarrow B) \in X_A$ and $[\forall](\exists B' \rightarrow B') \in X_A$. Hence, $C \vdash \exists B \land \exists B'$. By the nesting axiom, $C \vdash [\forall](\exists B' \rightarrow B' \lor (B' \rightarrow B))$. Therefore, either $V \subseteq V'$ or $V' \subseteq V$. \hfill \Box

**Definition 62.** A subset $Q$ of $T$ is sibling disjoint if for each $V, V' \in Q$, $V \cap V' = \emptyset$. Consider the inclusion $\subseteq$ on the set of sibling disjoint subsets of $T$. A subset $Q$ of $T$ is suitable iff it is a $\subseteq$-maximal sibling disjoint. For each suitable $Q \subseteq T$, the canonical partition of $Q$ is $Q \cup \{ \{ y \} : y \notin \bigcup Q \}$. A partition $P$ is canonical if it is the canonical partition of some suitable $Q$.

**Lemma 63.** For each $V \in T$ there is a suitable subset $Q$ of $T$ such that $V \in Q$. 
Proof. Pick $V \in T$ arbitrarily. Enumerate $T$ as $V = V_0, V_1, \ldots, V_n$, and let $Q_0 = \{V_0\}$. Define

$$Q_{i+1} = \begin{cases} Q_i \cup \{V_{i+1}\} & \text{if } V_{i+1} \cap \bigcup Q_i = \emptyset \\ Q_i & \text{otherwise} \end{cases}$$

Each $Q_i$ is sibling disjoint and $V \in Q_n$. Obviously $Q_i \subseteq Q_{i+1}$. Therefore, $Q_n$ is suitable and $V \in Q_n$ as desired. □

Lemma 64. Let $Q$ be a suitable subset $Q$ of $T$, let the canonical partition $P$ of $Q$. Then, $V \in Q$ iff there is $B$ such that $V = \{C : C \vdash B\} \in P$.

Proof. It is immediate from the definition that there is $B$ such that $V = \{C : C \vdash B\} \in P$ if $V \in T$. Conversely, if there were any $V$ such that there is $B$ such that $V = \{C : C \vdash B\} \in P$, $V \in Q$ since otherwise $Q$ cannot be suitable. □

For each $V \in T$ and each partition $P$ of $U_0$, let $S(P, V) = \{U \in P : U \cap V \neq \emptyset\}$.

Lemma 65. Let $P$ be a canonical partition on $U_0$, $V \in T$, and $U \in S(P, V)$. Suppose $U \subseteq V$. Then $\bigcup S(P, V) = V$.

Proof. Let us show that for an arbitrary $U \in P$, $U' \in S(P, V)$ iff $U' \subseteq V$. It is easy to see that $U' \subseteq V$ implies $U' \in S(P, V)$. For the converse, suppose $U' \not\subseteq V$. Then there is $x \in U' \setminus V$, which implies $U' \in T$. Then $V \subseteq U'$ must hold, while at the same time there is $U \in S(P, V)$ such that $U \subseteq V$. It follows that $U \not\subseteq U'$, hence $P$ cannot be a partition, against the assumption. Therefore, for each $U' \in P$, $U' \in S(P, V)$ iff $U' \subseteq V$. It implies that $\bigcup S(P, V) = V$. □

Lemma 66. Let $P$ be a canonical partition on $U_0$, $V \in T$, and $U \in S(P, V)$. Suppose $V \subseteq U$. Then $S(P, V) = \{U\}$.
Proof. Suppose $U' \in S(P, V)$. Then $U' \not\subseteq V$ must hold, as otherwise $U' \subseteq U$ must hold, which implies $P$ cannot be a partition. Hence, there is $x \in U' \setminus V$, thus $U'$ is not a singleton, so $U' \in T$. Therefore, $V \subseteq U'$ must hold. Moreover, $U = U'$, as $P$ is a partition. Therefore, $S(P, V)$ is a singleton. □

Proposition 67. Let $P$ be a canonical partition on $U_0$ and $V \in T$. Suppose there is $U \in S(P, V)$ such that $U \subseteq V$. Then there is a canonical partition $P'$ such that $P \leq P'$.

Proof. By Lemma 65, $\bigcup S(P, V) = V$ holds. It implies that $P' = (P \setminus S(P, V)) \cup \{V\}$ is a partition. Pick a suitable $Q \subseteq P$. Define $Q' = (Q \setminus S(P, V)) \cup \{V\} \subseteq T$ is also suitable, since otherwise $Q$ cannot be suitable. Let $P'$ be the canonical partition of $Q'$. Consider an arbitrary $U \in P$. If $U \in (P \setminus S(P, V))$, $U \in P'$. If $U \in S(P, V)$, $U \subseteq V \in P'$. Therefore, $P \leq P'$.

Proposition 68. Let $P$ be a canonical partition on $U_0$ and $V \in T$. Suppose there is $U \in S(P, V)$ such that $V \subseteq U$. Then there is a is a canonical partition $P' \leq P$.

Proof. By Lemma 66. $S(P, V)$ is a singleton $\{U\}$. Let $Q \subseteq P$ be a suitable partition. Define $Q' = (Q \setminus S(P, V)) \cup \{V\}$. Then, $Q'$ is suitable, for if not, $Q$ cannot be suitable either. Let $P'$ be the canonical partition of $Q'$.

Consider an arbitrary $U' \in P'$. If $U' \in S(P, V)$, $U' \subseteq U \in P$; if $U' \not\in S(P, V)$, $U' \in P$. Therefore, $P' \leq P$. □

Proposition 69. Let $P, P'$ be canonical partitions on $U_0$ such that $P \leq P'$, and $V \in T$. Suppose that there is $U \in S(P, V)$ such that $V \subseteq U$ and that there is $U' \in S(P', V)$ such that $V \subseteq U'$. Then there is a canonical partition $P''$ such that $P \leq P'' \leq P'$.

Proof. By Lemmas 65 and 66, $\bigcup S(P, V) = V$ and $S(P', V)$ is a singleton $\{U'\}$. 

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Define \( Q'' = Q' \setminus S(P', V) \cup \{V\} \cup Q \setminus S(P, V) \). Then \( Q'' \subseteq T \) and \( Q'' \) is suitable.

Let \( P'' \) be the canonical partition of \( Q'' \).

\( P \leq P'' \), by considering an arbitrary \( U \in P \). If \( U \in (P \setminus S(P, V)) \), \( U \in P'' \). If \( U \in S(P, V) \), \( U \subseteq V \in P'' \). Therefore, \( P \leq P'' \). Also \( P'' \leq P' \). by considering \( U'' \in S(P', V) \). \( U'' \subseteq U \in P' \); if \( U'' \not\in S(P', V) \), \( U'' \in P' \). Therefore, \( P'' \leq P' \). \( \square \)

**Lemma 70.** There is a \( \leq \)-chain \( \Pi \) of canonical partitions.

**Proof.** Pick \( V \in T \) arbitrarily. Enumerate \( T \) as \( V = V_0, V_1, \ldots, V_n \). Fix an arbitrary suitable subset \( Q_0 \) of \( U_0 \) such that \( V \in Q_0 \), and let \( P_0 \) be the canonical partition of \( Q_0 \). Let \( \Pi_0 = \{P_0\} \).

A chain of sets of canonical partitions is to be defined step by step. The construction in \( k + 1 \)'s step is as follows. Suppose \( \Pi_k \) is defined and its members are \( \leq \)-ordered as \( P_{k_0} \leq \cdots \leq P_{k_k} \). \( V_{k+1} \) is to be considered. Notice that, since every set \( V_i \) in \( T \) is non-empty, in each partition there must be a member overlapping with \( V_i \).

There are three cases.

(1) For each \( l \) (\( 0 \leq l \leq k \)) and for each \( V \in S(P_k) \), \( V \not\subset V_{k+1} \).

Define \( P_{k+1} = (P_k \setminus S(P_k, V_{k+1})) \cup \{V_{k+1}\} \). By Proposition 67, \( P_{k+1} \) is a canonical partition and \( P_{k_k} \leq P_{k+1} \).

Let \( \Pi_{k+1} = \Pi_k \cup \{P_{k+1}\} \) and extend the order \( P_{k_0} \leq \cdots \leq P_{k_k} \leq P_{k+1} \).

(2) For each \( l \) (\( 0 \leq l \leq k \)) and for each \( V \in S(P_k) \), \( V_{k+1} \not\subset V \). \( S(P_k) \) must be the singleton \( \{V\} \). Define \( P_{k+1} = (P_k \setminus S(P_k)) \cup \{V_k\} \cup \{x : x \in \bigcup S(P_k) \setminus V_k\} \).

By Proposition 68, \( P_{k+1} \) is a canonical partition and \( P_{k+1} \leq P_{k_0} \).

Let \( \Pi_{k+1} = \Pi_k \cup \{P_{k+1}\} \) and extend the order \( P_{k+1} \leq P_{k_0} \leq \cdots \leq P_{k_k} \).

(3) There is \( l \) (\( 0 \leq l \leq k \)) such that there is \( V \in P_l \) such that \( V \subset V_{k+1} \) and there is \( m \) (\( 0 \leq m \leq k \)) such that there is \( V' \in P_m \) such that \( V \subset V_{k+1} \).

Let \( P_l \) be the maximal among those such that there is \( V \in P_l \) such that
$V \subseteq V_{k+1}$ and $P^m$ be the minimal among those such that there is $V' \in P_{km}$ such that $V \subseteq V_{k+1}$.

$$P_{k+1} = P^m \setminus S(P^m) \cup \{V_{k+1}\} \cup P^l \setminus S(P^l)$$

By Proposition 69, $P_{k+1}$ is a canonical partition. Moreover, $P_K = P^l \leq P^m = P_{K+1}$, and $P^l \leq P_{k+1} \leq P^m$.

Let $\Pi_{k+1} = \Pi_k \cup \{P_{k+1}\}$ and extend the order $P_k \leq \cdots \leq P_l \leq P_{k+1} \leq P^m \leq \cdots \leq P_k$.

Repeat the construction up to $k = n - 1$, and let $\Pi = \Pi_n$. Obviously, $\Pi_n$ is a set of partitions linearly ordered with respect to $\leq$.

\[ \square \]

Let $v : Prop \to \varnothing U_0$ as $v(p) = \{C : C \vdash p\}$. Define $M_l = \langle U_0, \Pi, v \rangle$.

**Lemma 71.** $M_l, C \models B$ iff $C \vdash_S B$.

**Proof.** The cases where the outermost operator is not $\nabla$ are similar to Proposition 57.

Suppose $B = \nabla B_0$. First, suppose $C \vdash \nabla B_0$. Then $\langle \exists \rangle \nabla B_0 \in X_A$, so $B_0 \in \Omega$, thus $X = \{C : C \vdash B_0\} \in T$. Thus, by Proposition 70, there is $P \in \Pi$ such that $X \in P$. Therefore, $M_l, C \models B$.

Conversely, suppose $M_l, C \models \nabla B_0$. It is equivalent to that there is $P \in \Pi$ such that $X \in P$, and $C \in X$. Since each $P$ is canonical, $C \vdash \nabla B_0$.

\[ \square \]

**Corollary 72.** Suppose $A$ is valid in the class of linear $p$-frame. Then $\vdash_S A$.

**Proof.** Given a $S$-consistent $A$, there is $C \in M_l$ such that $M_l, C \models A$, since $A \in X_A$. Thus, there is a linear $p$-model of $A$.

\[ \square \]
9. Further direction

An intended interpretation of a linearly ordered set of partitions is a situation with multiple contexts where a message bears information in various levels of details, as seen in everyday life where ambiguous messages often appear. Impossibility of characterization of the class of frames a linearly ordered partition set, while being formally interesting, looks despairing for the application.

It seems necessary to extend either semantics or language, or even both, to characterize such a notion. Nevertheless, it is still open whether a similar class of frames with a linearly ordered partition set is characterized in a system of any language.
Logics of partial answers

In Chapter 2 Section 7, a further classification of partial answers is proposed. This section presents several operators concerning formal counterparts around partial answers. Logical relationships among those operators are also argued.

As mentioned on page 38, I propose to refine the notion of partial answer into three categories. First, partial but possibly not informative answers; second, partial and informative answers; and third, partial answers which are disjunctions of complete and just complete answers.

In this chapter, their formal counterparts are argued except the first notion which is represented by a standard S5 operator. [dstit] for the second and [pure] for the third are mainly focused.

1. $\langle \neq \rangle$

$\langle \neq \rangle$ operator\(^1\) can be seen as “loose informativeness,” or informativeness without veridicality. It cannot directly taken as an informativeness operator, since veridicality is often required to be an informative answer.

2. [dstit]

In the sense, the [dstit] operator seems more adequate to represent the notion of informativeness. In fact, it nicely reflects logical properties of partial answers.

\(^1\)See Chapter 5 Section 6.
Consider $L(\diamond, [\text{dstit}])$ and $L([\text{dstit}])$. An $L([\text{dstit}])$-frame is $\langle W, R_{[\text{dstit}]} \rangle$, where $R_{[\text{dstit}]}$ is an equivalence relation. A $L([\text{dstit}])$-valuation is defined as usual. A $L([\text{dstit}])$-model is $\langle W, R_{[\text{dstit}]}, v \rangle$ where $\langle W, R_{[\text{dstit}]}, v \rangle$ is an $L([\text{dstit}])$-frame and $v$ is an $L([\text{dstit}])$-valuation. The truth conditions for boolean operators are as usual.

$w \models [\text{dstit}]A$ iff (1) for every $w'$ such that $R_{[\text{dstit}]}ww'$, $w \models A$ and (2) there is $w''$ such that $w \not\models A$

Axiomatic systems in $L(\diamond, [\text{dstit}])$ and $L([\text{dstit}])$ for the logic on the structure are given in [71] and [72] with a direct completeness proof of each system.

3. Languages $L(\diamond, [\text{pure}])$ and $L([\text{pure}])$

A new modal operator for strict partial answer is to be introduced. The languages $L(\diamond, [\text{pure}])$ and $L([\text{pure}])$ are similar to $L(\diamond, [\text{dstit}])$ and $L([\text{dstit}])$. The truth condition is given as follows:

$w \models [\text{pure}]A$ iff (1) for every $w'$ such that $R_{[\text{pure}]}ww'$, $w \models A$

(2) there is $w''$ such that $w \not\models A$

(3) for every equivalence class $P$,

either $P \models A$ or $P \models \neg A$

**Lemma 73.** The following are equivalent: (a) There is $w \in W$ such that $w \models (\Box B \lor \Box \neg B) \land \neg [\text{pure}](\Box B \lor \Box \neg B)$; and (b) $W \models \Box B \lor \Box \neg B$.

**Proof.** It is trivial that (b) implies (a). By definition, $w \models \neg [\text{pure}](\Box B \lor \Box \neg B)$ is equivalent to either (1) $Rw \not\models \Box B \lor \Box \neg B$, (2) $W \models \Box B \lor \Box \neg B$ or (3) there is a mixed partition $P_*$ for $(\Box B \lor \Box \neg B)$, i.e., $P_* \models \diamond(\Box B \lor \Box \neg B) \land \neg(\Box B \lor \Box \neg B)$. But (3) contradicts with $\vdash_{SS} \diamond(\Box B \lor \Box \neg B) \rightarrow (\Box B \lor \Box \neg B)$. Thus (2).

**Corollary 74.** For some $w \in W$, $w \models \Box B \land \neg [\text{pure}](\Box B \lor \Box \neg B) \land \neg [\text{pure}][B$ iff $W \models B$. 

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PROOF. By the lemma, \( W \models \Box B \vee \Box \neg B \). It means that there is no mixed partition for \( B \). On the other hand, \( w \models \neg \text{[pure]} B \) is equivalent to either (1) \( Rw \not\models \Box B \), (2) \( W \models B \), or (3) there is mixed partition for \( B \). (1) and (3) are contradictory. Thus (2).

3.1. Languages \( L(\forall, \Diamond) \) and \( L(\forall) \). The truth condition of the universal operator \( \forall \) is defined as follows:

\[
\forall A \text{ iff for every } w' \in W, w \models A
\]

4. Definability

\( L(\langle \neq \rangle, \Diamond) \) is expressive. \([\text{dstit}] A \) can be defined as \( \Box A \land \langle \neq \rangle \neg A \), \([\text{pure}] A \) as \( \Box A \land \langle \neq \rangle \neg A \land \langle \neq \rangle (\Box A \leftrightarrow A) \).

\( \forall \) is defined in \( L(\Diamond, \text{[pure]}) \) as follows:

\[
(7) \quad \forall A \text{ iff } \Box A \land \neg \text{[pure]} A \land \neg \text{[pure]} (\Box A \lor \Box \neg A)
\]

Vice versa, \( \text{[pure]} \) is defined in \( L(\Diamond, \forall) \) as follows:

\[
(8) \quad \text{[pure]} A \text{ iff } \Box A \land \neg \forall A \land \forall (\Box A \leftrightarrow A)
\]

**Proposition 75.** \( \langle \neq \rangle \)-operator is not definable in \( L(\Diamond, \text{[pure]}) \).

PROOF. Consider two structures,

\[
M_1 = \langle \{w, x\}, \{\{w\}, \{x\}\}, v_1 \rangle
\]
\[
M_2 = \langle \{w_0, w_1, x\}, \{\{w_0, w_1\}, \{x\}\}, v_2 \rangle
\]
where \( v_1(p_0) = v_2(p_0) = x \) and \( v_1(p) = v_2(p) = \emptyset \) for any propositional variable \( p \) other than \( p_0 \). Then, for any formula which does not contain both \( p_0 \) and \([\text{pure}]\), these models support the same. But \( M_1, w \models \neg (\neq) p_0 \) while \( M_2, w_1 \models (\neq) p_0 \).

\[\Box\]

**Proposition 76.** \( \Diamond \) is not definable in \( L((\neq), [\text{pure}]) \).

**Proof.** Consider two structures:

\[
M_1 = \langle \{x, y, z, w\}, \{\{x, y, z\}, \{w\}\}, v_1 \rangle
\]

\[
M_2 = \langle \{x', y', z', w'\}, \{\{x', y'\}, \{z', w'\}\}, v_2 \rangle
\]

where \( v_1(p_0) = \{x, w\} \) and \( v_1(q) = \emptyset \) for any \( q \neq p_0 \), and \( v_1(p_0) = \{x', w'\} \) and \( v_1(q) = \emptyset \) for any \( q \neq p_0 \). Then, \( M_1, w \not\models \Box p_0 \) and \( M_2, w' \models \Box p_0 \), while they support the same \( L((\neq), [\text{pure}]) \)-formulas.

\[\Box\]

A model is pure for \( B \) iff for each equivalence class \( P \), \( P \not\models B \) or \( P \models \neg B \). Purity can be defined in each of \( L((\neq), \Diamond) \) and \( L((\neq), [\text{pure}]) \) as well as in \( L(\Diamond, [\text{pure}]) \) as above: defined as \( (\Box B \lor \Box \neg B) \land \neg [\text{pure}](\Box B \lor \Box \neg B) \). In fact, \( (\Box A \lor \Box \neg A) \land \neg (\neq) \neg (\Box A \lor \Box \neg A) \) and \( (\neq) \neg [\text{pure}] A \lor (A \land \neg (\neq) \neg A) \lor (\neg A \land \neg (\neq) A) \) express purity. Obviously it is not definable in \( L((\neq)) \).

\[\Box\]

\( L([\neq], \Box) \) is expressive. [dstit] \( A \) can be defined as \( \Box A \land (\neq) \neg A \), [pure] \( A \) as \( \Box A \land (\neq) \neg A \land [\neq](\Box A \leftrightarrow A) \).

\[\Box\]

**Lemma 77.** The following are equivalent: (a) There is \( w \in W \) such that \( w \models (\Box B \lor \Box \neg B) \land \neg [\text{pure}](\Box B \lor \Box \neg B) \); and (b) \( W \models \Box B \lor \Box \neg B \).

**Proof.** It is trivial that (b) implies (a). By definition, \( w \models \neg [\text{pure}](\Box B \lor \Box \neg B) \) is equivalent to either (1) \( Rw \not\models \Box B \lor \Box \neg B \), (2) \( W \models \Box B \lor \Box \neg B \) or (3) there is a
mixed partition $P_*$ for $(\Box B \lor \Box \neg B)$, i.e., $P_* \models \diamondsuit(\Box B \lor \Box \neg B) \land \diamondsuit \neg(\Box B \lor \Box \neg B)$. But (3) contradicts with $\vdash_{SS} \diamondsuit(\Box B \lor \Box \neg B) \rightarrow \Box(\Box B \lor \Box \neg B)$. Thus (2).

**Corollary 78.** For some $w \in W$, $w \models \Box B \land \neg\text{[pure]}(\Box B \lor \Box \neg B) \land \neg\text{[pure]}B$ iff $W \models B$.

**Proof.** By the lemma, $W \models \Box B \lor \Box \neg B$. This means that there is no mixed partition for $B$. On the other hand, $w \models \neg\text{[pure]}B$ is equivalent to either (1) $Rw \not\models \Box B$, (2) $W \models B$, or (3) there is mixed partition for $B$. (1) and (3) are contradictory. Thus (2). □

The lemma and corollary mean that $[\forall]$ is defined in $L(\Box, [\text{pure}])$ as:

(9) $[\forall]A$ iff $\Box A \land \neg[\text{pure}]A \land \neg[\text{pure}](\Box A \lor \Box \neg A)$

and vice versa, $[\text{pure}]$ is defined in $L(\Box, [\forall])$ as:

(10) $[\text{pure}]A$ iff $\Box A \land \neg[\forall]A \land [\forall](\Box A \leftrightarrow A)$

The expressive power of languages with $\langle \neq \rangle$ is strictly stronger than $L(\Box, [\text{pure}])$.

**Proposition 79.** $\langle \neq \rangle$-operator is not definable in $L(\Box, [\text{pure}])$.

**Proof.** Consider two structures,

$M_1 = \langle \{w, x\}, \{\{w\}, \{x\}\}, v_1 \rangle$

$M_2 = \langle \{w_0, w_1, x\}, \{\{w_0, w_1\}, \{x\}\}, v_2 \rangle$

where $v_1(p_0) = v_2(p_0) = x$ and $v_1(p) = v_2(p) = \emptyset$ for any propositional variable $p$ other than $p_0$. Then, for any formula which does not contain both $p_0$ and $[\text{pure}]$, these models support the same. But $M_1, w \not\models \langle \neq \rangle p_0$ while $M_2, w_1 \models \langle \neq \rangle p_0$. □
Nevertheless, $\langle \neq \rangle$ cannot define $\Diamond$.

**Proposition 80.** $\Diamond$ is not definable in $L(\langle \neq \rangle, [\text{pure}])$.

**Proof.** Consider two structures:

$$M_1 = \langle \{x, y, z, w\}, \{\{x, y, z\}, \{w\}\}, v_1 \rangle$$

$$M_2 = \langle \{x', y', z', w'\}, \{\{x', y'\}, \{z', w'\}\}, v_2 \rangle$$

where $v_1(p_0) = \{x, w\}$ and $v_1(q) = \emptyset$ for any $q \neq p_0$, and $v_1(p_0) = \{x', w'\}$ and $v_1(q) = \emptyset$ for any $q \neq p_0$. Then, $M_1, w \not\models \Box p_0$ and $M_2, w' \models \Box p_0$, while they support both of neither of these $L(\langle \neq \rangle, [\text{pure}])$-formulas. $\square$

Definability is summarized in the table; $\checkmark$ means the operator on the top of the column is definable in the language on the row; $\times$ means undefinability; and $-$ means the operator is already in the language.

<table>
<thead>
<tr>
<th></th>
<th>$\langle \neq \rangle$</th>
<th>$[\forall]$</th>
<th>[pure]</th>
<th>$\Box$</th>
<th>‘purity’</th>
<th>$[\text{dstit}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L([\neq])$</td>
<td>$-$</td>
<td>$\checkmark$</td>
<td>$\times$</td>
<td>$\times$</td>
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<tr>
<td>$L(\Box, [\text{pure}])$</td>
<td>$\times$</td>
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<td>$\checkmark$</td>
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<tr>
<td>$L([\neq], [\text{pure}])$</td>
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<td>$L(\Box, [\forall])$</td>
<td>$\times$</td>
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</tr>
<tr>
<td>$L([\text{pure}], [\forall])$</td>
<td>$\times$</td>
<td>$-$</td>
<td>$-$</td>
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<td>$L(\Box, [\neq])$</td>
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<td>$\checkmark$</td>
</tr>
<tr>
<td>$L([\text{dstit}])$</td>
<td>$\times$</td>
<td>$\checkmark$</td>
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<td>$\checkmark$</td>
<td>$-$</td>
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</tr>
</tbody>
</table>

For comparison with other operator, consider p-frames with a single partition in its partition set, and let $[\text{unique}]_{\{\}}$ be the [unique] operator in such semantics.
In fact, \([\text{unique}]_{\{\}}\) is expressive. As mentioned in Chapter 6 Section 6, each of the following semantical conditions correspond to an additional axiom in the language \(L([\text{unique}]_{\{\}})\), while none can be in \(L([\neq])\):

(1) At-least-\(n\)-equivalence-classes;

\[
\bigwedge_{0 \leq i \leq n-1} (\exists)\,[\text{unique}]_{\{\}} ((\bigwedge_{0 \leq j \leq n-1} \neg A_j) \land A_i) \to (\exists)\,\bigwedge_{0 \leq i \leq n-1} \neg A_i
\]

(2) Exactly-1-equivalence-class: \([\text{unique}]_{\{\}} \top\)

(3) Not-2-equivalence-classes (either 1 or more than 2):

\([\text{unique}]_{\{\}} p \to [\forall] [\neg [\text{unique}]_{\{\}} \neg p]

When there is more than one equivalence class in the partition, it holds that \([\text{unique}]_{\{\}} A \to [\text{pure}] A\). In particular, when there are exactly two equivalence classes, \([\text{unique}]_{\{\}} A\) and \([\text{pure}] A\) are semantically equivalent.

Considering the above property, it is natural to see that \([\text{unique}]_{\{\}}\) operator is not definable in \(L([\neq], \square)\); in fact, compare the following two models:

\(M_1 = \langle \{x_1, x_2, x_3\}, \{\{x_1\}, \{x_2\}, \{x_3\}\}, v_1 \rangle\)

\(M_2 = \langle \{y_1, y_2, y_3\}, \{\{y_1\}, \{y_2\}, \{y_3\}\}, v_2 \rangle\)

\(v_1(p) = \{x_1, x_2\}, \quad v_2(p) = \{y_1, y_2\}\)

\(v_1(q) = v_2(q) = \emptyset\) if \(p \neq q\)

Then \(M_1, x_1 \not\models [\text{unique}]_{\{\}} p\) and \(M_2, y_1 \models [\text{unique}]_{\{\}} p\) while they agree on \(L([\neq], \square)\) formulas.

Vice versa, \([\neq]\) is not definable in \(L([\text{unique}]_{\{\}}, \square)\): for

\(M_1 = \langle \{x_1, x_2, x_3\}, \{\{x_1\}, \{x_2\}, \{x_3\}\}, v_1 \rangle\)

\(M_2 = \langle \{y_1, y_2\}, \{\{y_1\}, \{y_2\}\}, v_2 \rangle\)
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\[ v_1(p) = \{x_1, x_2\}, \quad v_2(p) = \{y_1\} \]

\[ v_1(q) = v_2(q) = \emptyset \text{ if } p \neq q \]

Then \( M_1, x_1 \models (\neq)p \) and \( M_2, y_1 \not\models (\neq)p \) while they agree on \( L([\text{unique}],[\Box]) \) formulas.

\([\forall] \) is not definable in \( L([\text{unique}],[\Box]) \) in general, but only on the structures with a trivial partition: obviously \([\text{unique}][\Box]A \leftrightarrow [\forall]A \).

5. Mutual translation between \( L(\Diamond, [\forall]) \) and \( L(\Diamond, [\text{pure}]) \)

Mutual translation between \( L(\Diamond, [\forall]) \) and \( L([\text{distit}]) \) is suggested in literature.

Define translations \( \tau \) from \( L(\Diamond, [\forall])-\)formulas to \( L(\Diamond, [\text{pure}])-\)formulas and \( \tau^{-1} \) from \( L(\Diamond, [\text{pure}])-\)formulas to \( L(\Diamond, [\forall])-\)formulas as follows.

\[
\begin{align*}
\tau(p) & \text{ iff } p \\
\tau(\neg A) & \text{ iff } \neg \tau(A) \\
\tau(A \land B) & \text{ iff } \tau(A) \land \tau(B) \\
\tau([\Box]A) & \text{ iff } [\Box]\tau(A) \\
\tau([\forall]A) & \text{ iff } [\forall]\tau(A) \\
\tau^{-1}(p) & \text{ iff } p \\
\tau^{-1}(\neg A) & \text{ iff } \neg \tau^{-1}(A) \\
\tau^{-1}(A \land B) & \text{ iff } \tau^{-1}(A) \land \tau^{-1}(B) \\
\tau^{-1}(\Box A) & \text{ iff } \Box \tau^{-1}(A) \\
\tau^{-1}([\text{pure}]A) & \text{ iff } \text{pure}(\tau^{-1}(A) \land [\forall](\Box \text{pure}(A) \leftrightarrow \tau^{-1}(A)))
\end{align*}
\]

The following is immediate from the truth condition of \([\text{pure}]\).

**Proposition 81.** For any \( L(\Diamond, [\text{pure}])-\)formula \( A, w \models A \) iff \( w \models \tau^{-1}(A) \).

For \( \tau \), the following follows from the lemma.

**Proposition 82.** For any \( L(\Diamond, [\forall])-\)formula \( A, w \models A \) iff \( w \models \tau(A) \).
6. Axiomatic system

Abbreviation: $\forall A \equiv \square A \land \neg\text{[pure]} A \land \neg\text{[pure]}(\square A \lor \square \neg A)$.

(1) S5 for $\square$
(2) $\text{[pure]} A \rightarrow \text{[pure]}\text{[pure]} A$
(3) $\text{[pure]} A \rightarrow \neg\text{[pure]}(\square A \lor \square \neg A)$ (purity axiom)
(4) $[\text{pure}] A \rightarrow \square A$
(5) $\Diamond[\text{pure}] A \rightarrow [\text{pure}] A$
(6) $\neg\text{[pure]} \neg[\forall] A$
(7) $A \leftrightarrow B/[\text{pure}] A \leftrightarrow [\text{pure}] B$

Provable formulas

- $\neg\text{[pure]} \top$ (Put $\top$ in purity axiom.)
- $\neg\text{[pure]} \bot$ ($\text{[pure]} A \rightarrow \square A$ and $\top$ for $\square$)
- $[\forall] A \rightarrow \square A$ (def $[\forall]$)
- $\Diamond[\forall] A \rightarrow [\forall] A$ (def $[\forall]$), 4 for $[\text{pure}]$
- $\Diamond[\forall] A \leftrightarrow [\forall] A$
- $\square[\forall] A \leftrightarrow [\forall] A$ (def $[\forall]$, $\Diamond[\text{pure}] A \rightarrow [\text{pure}] A$)
- $\Diamond[\forall] A \leftrightarrow \square[\forall] A$
- $\neg\text{[pure]}(\square[\forall] A \lor \square \neg[\forall] A)$ ($\square \neg[\forall] A \leftrightarrow \neg\square[\forall] A$ and $\neg\text{[pure]} \top$)

In particular: We can have S5 for $[\forall]$ from the above axioms.

- $[\forall] A \rightarrow A$
- $\neg[\forall] \neg[\forall] A \rightarrow [\forall] A$
- $A/[\forall] A$

**Proposition 83.** $[\forall] A \rightarrow A$ is a theorem.

**Proof.** Definition and A1. $\square$
Proposition 84. \( \neg \forall \neg \forall A \rightarrow [\forall]A \) is a theorem.

Proof. By definition of \( [\forall] \), \( \neg \forall \neg [\forall]A \rightarrow [\forall]A \lor [\text{pure}] \neg [\forall]A \lor [\text{pure}] (\square [\forall]A \lor \square \neg [\forall]A) \). Since \( [\forall]A \leftrightarrow [\forall]A \land [\forall]A \lor [\text{pure}] \neg [\forall]A \), and \( \neg [\forall]A \lor [\text{pure}] \neg [\forall]A \) are theorems, \( \neg [\forall]A \neg [\forall]A \rightarrow [\forall]A \) as desired. \( \square \)

Proposition 85. \( [\forall]A \) is a derivable rule.

Proof. When \( A \) is provable, so is \( [\forall]A \). By definition, \( [\forall]A = [\forall]A \land [\text{pure}] [\forall]A \land [\text{pure}] (\square [\forall]A \lor \square \neg [\forall]A) \). The first conjunct is a theorem by A1, while, as \( \square [\forall]A \lor \square \neg [\forall]A \leftrightarrow \top \) and \( \neg [\text{pure}] \top \) is a theorem, the last two conjuncts are theorems. Thus, \( [\forall]A \) is provable. \( \square \)

Thus, the construction with multi-S5 \( [\forall] \) and \( \square \) work for the above axiomatic system.

The following list is the valid formulas.

1. \( \neg [\text{pure}] (A \land \neg [\text{pure}]A) \)
2. \( [\text{pure}]A \land [\text{pure}]B \rightarrow [\text{pure}] (A \land B) \)
3. \( [\text{pure}]A \rightarrow [\text{pure}] \square A \)
4. \( [\text{pure}]A \land [\forall]B \rightarrow [\text{pure}] (A \land B) \)
5. \( [\text{pure}]A \land [\forall]B \rightarrow [\text{pure}] (A \land [\forall]B) \)
6. \( [\text{pure}] (A \land [\forall]B) \rightarrow [\text{pure}]A \land [\forall]B \)
7. \( \neg [\text{pure}] [\forall]A \) (But follows from \( [\text{pure}] (A \land [\forall]B) \rightarrow [\text{pure}]A \land [\forall]B) \)

7. Independence with \( L(\square, [\text{pure}]) \) and \( L(\square, [\text{dstit}]) \)

It follows that the logic for \( L(\square, [\text{pure}]) \) is independent from \( L(\square, [\text{dstit}]) \) (so called dstit).

In fact, \( [\text{dstit}]A \rightarrow \neg [\text{dstit}] (\square A \lor \square \neg A) \) is invalid in \( L(\square, [\text{dstit}]) \). Here is a countermodel: \( \langle \{x, y, z\}, \{\{x\}, \{y, z\}\}, v \rangle \), where \( v(p) = \{x, y\} \). Then, \( x \models [\text{dstit}]A \)
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but $x \models [\text{dstit}](\Box A \lor \Box \neg A)$. On the other hand, $[\text{pure}](B \land A \land \neg [\text{pure}] A) \rightarrow [\text{pure}] B$ is invalid, which is valid in $L(\Box, [\text{dstit}])$-semantics. Here is a countermodel: $x \not\models [\text{pure}](B \land A \land \neg [\text{pure}] A) \rightarrow [\text{pure}] B$ in $\langle \{x, y, z\}, \{\{x\}, \{y, z\}\}, v \rangle$, where $v(p_0) = \{x, y\}$, $v(p_1) = \{x, z\}$, and $v(p) = \emptyset$ for any $p \neq p_0, p_1$.

8. Chapter summary

This chapter has discussed relationships among logics of partial answers. They behave differently, and so do their informal counterparts proposed in Chapter 2.

Here is open problems:

- Show completeness: $\Sigma \vdash [\forall], \Box A$ iff $\Sigma \models_{\text{multi-S5}} A$ in $L(\Diamond, [\forall])$.
- Is there any translation from proofs in one language to those in another language?
Bibliography


Critical notes.


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